

Math 412 Matrix-tree theorem

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* See the handout *

Spanning tree directed into s : subgraph containing all the vertices such that
outdegree = 1 \forall vertices except s and
 $\text{outdeg}(s) = 0$.

K_{n+1} = complete directed graph on $n+1$ vertices $V = \{v_0, \dots, v_n\}$.

The edge set for K_{n+1} is $V \times V$ (thus, there are loops at each vertex).

Let x_{ij} be an indeterminate for $i, j \in \{0, \dots, n+1\}$. Think of these as generic weights on the edges of K_{n+1} .

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(generic) Laplacian matrix $(n+1) \times (n+1)$ matrix L

$$L_{ij} = \begin{cases} \sum_{k \neq i} x_{ik} & \text{for } i=j \\ -x_{ij} & \end{cases}$$

Thus, $L \mathbf{1} = \mathbf{0}$, where $\mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$. Hence, $\det L = 0$.

T_k = all spanning trees of K directed into v_k

If $\tau \in T_k$, defined its weight by

$$\text{wt}(\tau) = \sum_{(v_i, v_j) \in E(\tau)} x_{ij}$$

where $E(\tau)$ = edges of τ .

For each k , let $L^{(k)}$ denote the matrix obtained from L by omitting the k^{th} row and column of L . ③

Thm. (Matrix-tree) For each k

$$\det L^{(k)} = \prod_{\tau \in T_k} \text{wt}(\tau).$$

Warm-up

First of all, consider the $n \times n$ matrix $X = (x_{ij})_{1 \leq i,j \leq n}$.
A monomial in the expansion of $\det X$ has the form

$$x_{1\pi(1)} \cdots x_{n\pi(n)}$$

for some permutation π on $\{1, \dots, n\}$.

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Consider the subgraph of K_n determined by this monomial.

What does it look like? For instance, what if $n=9$ and

$$\pi = (2, 4, 9)(1, 8, 6, 3).$$



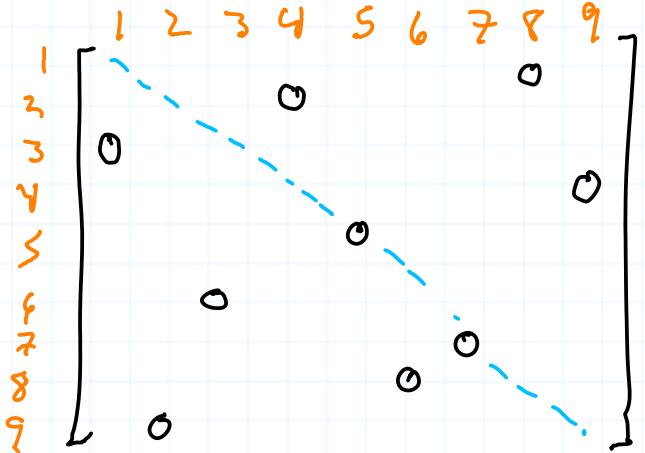
Harder question To which subgraphs of K_{n+1} do the monomials in the expansion of $\det L^{(0)}$ correspond?

Note:

$$\det L^{(0)} = \sum_{\pi \in S_n} \text{sgn}(\pi) L_{1\pi(1)} \cdots L_{n\pi(n)}$$

Consider the case $n=9$.

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picturing the term

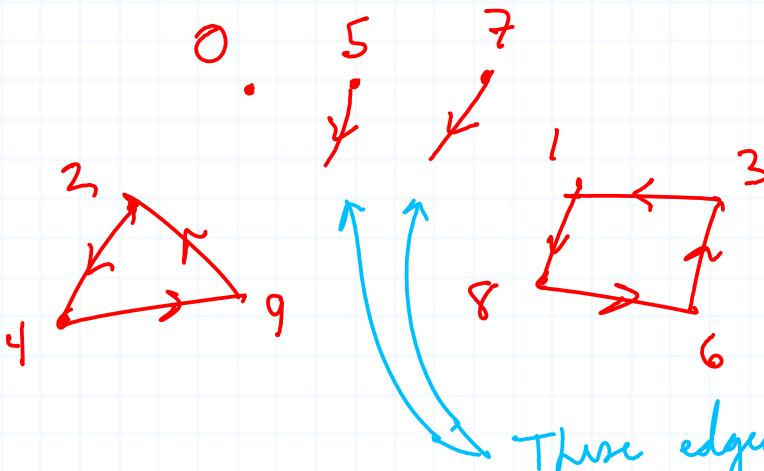
$$\text{sgn}(\pi) L_{1\pi(1)} \cdots L_{n\pi(n)}$$

$$\text{for } \pi = (\underbrace{2, 4, 9}_{\text{even cycle}}) (\underbrace{1, 8, 6, 3}_{\text{odd cycle}})$$

$$\left[(-x_{24})(-x_{49})(-x_{92}) \right] \left[(-x_{18})(-x_{86})(-x_{63})(-x_{31}) \right] \left[\sum_{\substack{k=0 \\ k \neq 5}}^1 x_{5k} \right] \left[\sum_{\substack{k=0 \\ k \neq 7}}^1 x_{7k} \right]$$

$\xrightarrow{-1}$ for even cycle $\xrightarrow{+1}$ odd cycle

But $\text{sgn}(\text{odd cycle}) = -1$, so
the "overall" sign is $(-1)^{\# \text{ of cycles}}$

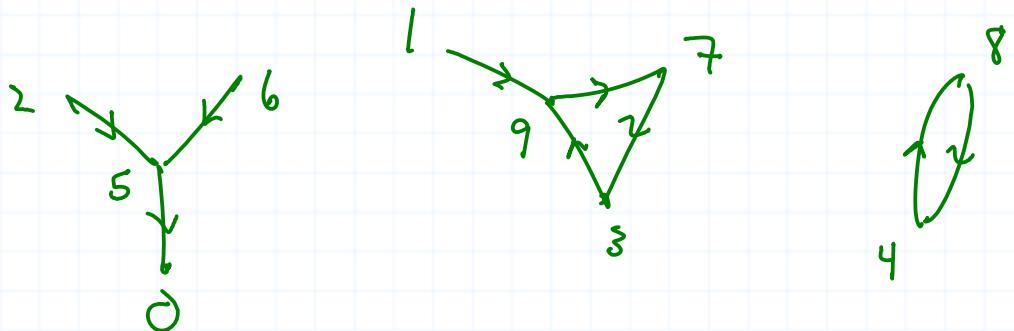


These edges are arbitrary (non-loops)

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So the monomials in the expansion of $\text{sgn}(\pi) L_{1\pi(1)} \cdots L_{n\pi(n)}$ for $\pi = (3, 4, 9)(1, 8, 6, 3)$ correspond to all subgraphs of K_{n+1} containing the cycles  and for which each vertex has outdegree 1, except for v_0 , which has outdegree 0.

Next question Conversely, letting $n = 9$ again, which permutations $\pi \in S_9$ give rise to the monomial $x_{19} x_{25} x_{39} x_{48} x_{50} x_{65} x_{73} x_{84} x_{97}$ and with what sign does the monomial appear?



Answer: The monomial appears for each $\pi \in S_9$ whose cycles are a

subset of $(3,9,7)$ and $(4,8)$.

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Possibilities for π

$$(3,9,7)(4,8)$$

$$(3,9,7)$$

$$(4,8)$$

$$()$$

Sign of $x_{19} x_{25} x_{39} x_{48} x_{50} x_{65} x_{73} x_{84} x_{97}$

$$(-1)^{\text{number of cycles}}$$

$$(-1)^1$$

$$(-1)^1$$

$$(-1)^0$$

binomial theorem

$$\text{sum} = \overline{\left(\binom{2}{2} (-1)^2 + \binom{2}{1} (-1)^1 + \binom{2}{0} (-1)^0 \right)} = (1-1)^2$$

choose 2 of 2 cycles choose 1 of two cycles choose 0 of 2 cycles

In this way, once cancellation occurs, the only monomials left in the expansion of $\det L^{(0)}$ are those corresponding

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to subgraphs of K_{n+1} containing all the vertices, with outdegree 1 for all vertices except v_0 , which has outdegree 0, and containing no cycles. In other words, only monomials corresponding to spanning trees directed into v_0 .

These monomials arise in the term of $\sum_{\pi} \text{sgn}(\pi) L_{1\pi(1)} \cdots L_{n\pi(n)}$ corresponding to $\pi = ()$, i.e., to

$$L_1 \cdots L_n.$$

The monomials for each spanning tree occur in the expansion of this term exactly once, each with coefficient +1.