

Notation. Let $\mathcal{L} = \text{image}(\Delta: \mathbb{Z}V \rightarrow \mathbb{Z}V)$ and $\tilde{\mathcal{L}} = \text{image}(\tilde{\Delta}: \mathbb{Z}\tilde{V} \rightarrow \mathbb{Z}\tilde{V})$.

Thm. $S(\Gamma, s)$ is a group under stalk addition and

$$\begin{aligned} S(\Gamma, s) &\rightarrow \mathbb{Z}\tilde{V}/\tilde{\mathcal{L}} \\ c &\mapsto \tilde{c} \end{aligned}$$

← Goal for today.

is an isomorphism.

Lemma 1. Each element of $\mathbb{Z}\tilde{V}/\tilde{\mathcal{L}}$ is represented by a recurrent configuration.

PF/ Let $a = c_{\max} + \vec{1}$ (adding one grain of sand to c_{\max} at each vertex), and let $b = a - a^s$. Then $b_v > 0 \quad \forall v \in \tilde{V}$ and $b \equiv 0 \pmod{\tilde{\mathcal{L}}}$. Given any configuration c , take $k \gg 0$ so that

$(c+kb)_v \geq c_{\max, v}$. Then $c = (c+kb)^\circ \bmod \tilde{\Gamma}$ * and $(c+kb)^\circ$ is recurrent since it is reachable by adding sand to c_{\max} and stabilizing:

$$(c+kb)^\circ = ((c+kb - c_{\max}) + c_{\max})^\circ.$$

a configuration,
i.e., an element of $N\tilde{\Gamma}$.

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Lemma 2. Let $a = c_{\max} + \vec{1}$ and $b = a - a^\circ$. If c is recurrent, then $(b+c)^\circ = c$.

Pf/ Let c be recurrent. Then \exists config. d such that $(d+a)^\circ = c$
Then

$$d+a+b = d+a+a-a^\circ \xrightarrow{\text{valid}} d+a+a^\circ - a^\circ = d+a \rightarrow (d+a)^\circ = c$$

and

The arrow represents a valid sequence of vertex firings.

$$d+a+b \rightarrow (d+a)^\circ + b = c+b \rightarrow (c+b)^\circ.$$

(2)

* Recall that if $c \rightsquigarrow \tilde{c}$ by firing v , then $\tilde{c} = c - \tilde{\Delta}v$.

The result follows by uniqueness of stabilization. (3) \square

Lemma 3. There is a unique recurrent configuration in each equivalence class of $\mathbb{Z}^{\tilde{V}}$ modulo \tilde{L} .

Pf/ Let c' and c'' be recurrents and suppose $c' \equiv c'' \pmod{\tilde{L}}$.

Then

$$c' = c'' + \sum_{v \in \tilde{V}} n_v \tilde{\Delta}_v$$

for some $n_v \in \mathbb{Z}$. Let $J_- = \{v : n_v < 0\}$ and $J_+ = \{v : n_v > 0\}$, and write

$$c := c' + \sum_{v \in J_-} (-n_v) \tilde{\Delta}_v = c'' + \sum_{v \in J_+} n_v \tilde{\Delta}_v.$$

Let $a = c_{\max} + \tilde{l}$ and $b = a - a^\circ$. Take $h \gg 0$ so that

$$(c + kb)_v \geq \max_{v \in \tilde{V}} \{ |n_v| \text{outdeg}(v)\}.$$

Thus, each vertex v of $c+kb$ can be legally fired $|n_v|$ times. (4)

Therefore,

$$c+kb = c' + \sum_{v \in -} (-n_v) \bar{\Delta}_v + kb \rightarrow c' + kb \xrightarrow{\text{By lemma 2}} c'$$

and, similarly,

$$c+kb \rightarrow c''.$$

By uniqueness of stabilization, $c' = c''$. \square

Pf. of thm / The mapping

$$\begin{aligned} S(T, s) &\rightarrow \mathbb{Z}^T / \mathbb{Z} \\ c &\mapsto \bar{c} \end{aligned}$$

respects addition. It is surjective by lemma 1 and injective by lemma 2. \square

(5)

Formula that gives an algorithm for computing
the identity of $S(\Gamma, s)$:

$$\text{identity} = [2c_{\max} - (2c_{\max})^{\circ}]^{\circ}.$$

Note that the right-hand side is recurrent since

$$(2c_{\max} - (2c_{\max})^{\circ})^{\circ} = \left[[c_{\max} - (2c_{\max})^{\circ}] + c_{\max} \right]^{\circ}$$

$\underbrace{}_{\geq 0}}$

and, as an element of $\mathbb{Z}\bar{V}/\bar{\mathbb{Z}}$, it's equal to $\vec{0}$. The isomorphism $S(\Gamma, s) \rightarrow \mathbb{Z}\bar{V}/\bar{\mathbb{Z}}$ sends the identity of $S(\Gamma, s)$ to the equivalence class of $\vec{0}$ in $\mathbb{Z}\bar{V}/\bar{\mathbb{Z}}$, and there is a unique recurrent configuration in that equivalence class.