

Last time The **Tutte polynomial** for a finite undirected multigraph G is defined recursively by:

1. $T(\emptyset; x, y) = 1.$

2. If e is a loop, $T(G; x, y) = x T(G - e; x, y).$

3. If e is a bridge, $T(G; x, y) = x T(G/e; x, y).$

4. If e is neither a loop nor a bridge, then

$$T(G; x, y) = T(G - e; x, y) + T(G/e; x, y).$$

From now on, assume G is connected.

Def. Fix a sink s for G . If $c \in S(G, s)$, define the **level** of c by

$$\text{level}(c) = \deg(c) - \#(E) + \deg(s).$$

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Prop. For all $c \in S(G, s)$, we have

$$0 \leq \text{level}(c) \leq g = \#(E) - \#(V) + 1.$$

Also, $\text{level}(c_{\max}) = g$ and $\exists c \in S(G, s)$ s.t. $\text{level}(c) = 0$.

Pf/ HW.

Note: For each i s.t. $0 \leq i \leq g$, there exists $c \in S(G, s)$ s.t.

$\text{level}(c) = i$. Just add sand to minimal recurrents, building up to c_{\max} .

Thm. (Merino, 1999) $T(G; 1, y) = \sum_{i=0}^g r_i y^i =: P_s(y)$

where r_i is the number of recurrent configurations on (G, s) with level i . [If G has one or no vertices, we consider G to have one (empty) recurrent configuration and take its degree to be

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o (hence, its level to be $\#(E)$.)]

Pf/ We prove this by induction on the number of edges. If G has no edges, it has at most one vertex since it is connected. In this case $T(G, x, y) = 1$ and $P_s(y) = 1$.

Now suppose G has at least one edge. Let e be an edge connected to the sink, s . There are three case to consider:

I. e is a loop. In this case there is an isomorphism

$$S(G, s) \rightarrow S(G - e, s)$$

$$c \mapsto \tilde{c}$$

where $\tilde{c} = c$. Further,

$$\begin{aligned} \text{level}(\tilde{c}) &= \deg \tilde{c} - \# E(G - e) + \deg_{G - e} s \\ &= \deg c - (\# E - 1) + (\deg_G s - 2) \end{aligned}$$

$$= \text{level}(c) - 1.$$

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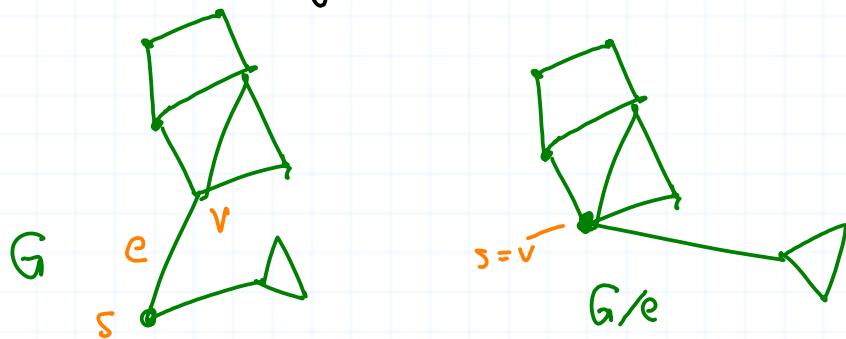
Therefore, $P_s(G; y) = y P_s(G-e; y)$. By induction, then

$$\begin{aligned} P_s(G-e; y) &= T(G-e; l, y), \text{ and thus, } P_s(G; y) = y P_s(G-e; y) \\ &= y T(G-e; l, y) = T(G, l). \end{aligned}$$

II. e is a bridge. By the burning algorithm, a configuration c on G }
is recurrent iff after firing the sink, c becomes unstable and in }
stabilizing, each (non sink) vertex fires exactly once. Let $e = (s, v)$. ★

It follows, in our case, that every recurrent configuration has
 $\deg(v) - 1$ grains of sand. We claim there is a bijection

$$\begin{array}{ccc} S(G, s) & \longrightarrow & S(G/e, s) \\ c & \mapsto & \tilde{c} \end{array}$$



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where $\tilde{c}(w) = c(w)$ for $w \in V(G) \setminus \{s, v\}$. The inverse is given by

$$c(w) = \begin{cases} \tilde{c}(w) & \text{for } w \in V(G) \setminus \{s, v\} \\ \deg(v)-1 & \text{for } w = v. \end{cases}$$

To see that c is recurrent iff $\tilde{c}_{\wedge}^{\text{is recurrent}}$, use the burning algorithm as it is described above (*). First suppose c is recurrent. Firing the sink in G adds one grain of sand to v , which can then fire, delivering one grain of sand to each of its neighbors; and subsequently the remaining vertices of G fire. In \tilde{G} the vertices s and v have been identified. Starting with \tilde{c} on \tilde{G} , firing the sink of \tilde{G} delivers one grain of sand directly to each of the neighbors of v . The subsequent firing sequence can then be exactly as it was for c .

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Similarly, if \tilde{c} is recurrent, so is c .

Now check the levels :

$$\begin{aligned}
 \text{level}(\tilde{c}) &= \deg \tilde{c} - \# E(G/e) + \deg_{G/e} s \\
 &= [\deg c - \deg v + 1] - [\# E - 1] + [\deg v - 1 + \deg_G s - 1] \\
 &= \deg c - \# E + \deg_G s \\
 &= \text{level}(c)
 \end{aligned}$$

Thus, $P(G; y) = P(G/e; y)$. By induction,

$$P(G; y) = P(G/e; y) = T(G/e; l, y) = T(G; l, y).$$

III e is neither a loop nor a bridge.

Let $e = (s, v)$. Divide the currents on G_0 into two sets: let A be those currents with $\deg v - 1$ grains on v , and let A' be the rest of the currents.

There is a bijection between A and the currents on G/e given just as in case II, again preserving levels.

There is a bijection between A' and the currents on $G \setminus e$ given by

$$A' \rightarrow S(G \setminus e, s)$$

$$c \mapsto \tilde{c}$$

where $\tilde{c}(w) = c(w) \quad \forall w \in V \setminus \{s\}$. To see this gives a bijection of currents note that when s fires in $G \setminus e$, the vertex v

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receives 1 fewer grains of sand than it would in G ,

however, $\deg_{G-e}(v) = \deg_G v - 1$ in compensation (as Mirine says).

The levels are related as follows

$$\begin{aligned}\text{level}(\tilde{c}) &= \deg \tilde{c} - \# E(G-e) + \deg_{G/S} s \\ &= \deg c - [\# E - 1] + \deg_G s - 1 \\ &= \deg c - \# E + \deg_G s \\ &= \text{level}(c).\end{aligned}$$

Therefore, using induction:

$$\begin{aligned}P_G(y) &= P_{G/e}(y) + P_{G-e}(y) = T(G/e; l, y) + T(G-e; l, y) \\ &= T(G; l, y).\end{aligned}$$