

Def. A **matroid** is a pair $M = (E, \mathcal{I})$ consisting of a finite set E and a collection of subsets \mathcal{I} of E called the **independent sets** of the matroid satisfying

0) $\emptyset \in \mathcal{I}$.

1) \mathcal{I} is closed under the operation of taking subsets:

$$I \in \mathcal{I} \text{ and } J \subseteq I \Rightarrow J \in \mathcal{I}.$$

2) The exchange axiom:

$$I, J \in \mathcal{I} \text{ and } |I| > |J| \Rightarrow \exists e \in I \setminus J \text{ s.t. } J \cup \{e\} \in \mathcal{I}.$$

Examples

1) Let V be a vector space, $E \subseteq V$, and let \mathcal{I} be the linearly independent subsets of E .

2) Let G be an undirected graph with edge set E , and let \mathcal{I} be the collection of forests of G (all acyclic subsets). A matroid arising from a graph is called **graphic**.

Vocabulary

The exchange property says that if $A \subseteq E$, then every maximal subset of A under inclusion has the same cardinality, $r(A)$, called the

rank of A . The rank of \mathcal{M} is $r(E)$. A subset of E of rank $r(\mathcal{M})$ is called a **basis** for \mathcal{M} .

(If \mathcal{M} is graphic, then $r(A) = \#(V) - k(A)$, where $k(A) = \#$ components of A .)

A **circuit** of \mathcal{M} is a minimal dependent (i.e., not independent) set. A **flat** is a subset $F \subseteq E$ such that, $\forall e \in E$

$$r(F \cup \{e\}) = r(F) \Rightarrow e \in F.$$

A **hyperplane** is a maximal proper flat.

★ A matroid can be characterized in terms of bases, circuits, flats, or hyperplanes.

The **dual** of \mathcal{M} is the matroid \mathcal{M}^* on the same set E but whose bases are defined to be the complements of bases of \mathcal{M} .

A **loop** of \mathcal{M} is an element $e \in E$ contained in no basis, i.e.

$r(\{e\}) = 0$. A **coloop** or **bridge** is an element $e \in E$ contained in every basis, i.e., a loop of \mathcal{M}^* .

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If e is not a loop, we can form a new matroid by contracting e :

$$M/e = (E \setminus \{e\}, \{I \setminus \{e\} : I \in \mathcal{I}\}).$$

Dually, if e is not a coloop, we get a new matroid by deleting e :

$$M \setminus e = (E \setminus \{e\}, \{I \in \mathcal{I} : e \notin I\}).$$

Example If M is a graphic matroid, a coloop is an edge whose deletion would increase the number of components of G . The matroids M/e and $M \setminus e$ come from contracting and deleting e in the usual sense.

Prop. If e is not a loop, $(M/e)^* = M^* \setminus e$.

⑤

Def. The **Tutte polynomial** is

$$T(x, y) = T(\mathcal{M}; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{\#(A)-r(A)}.$$

Thm. The Tutte polynomial is defined recursively by

1. $T(\emptyset; x, y) = 1$ where \emptyset is the empty matroid.
2. If e is a loop, $T(\mathcal{M}; x, y) = y T(\mathcal{M} \setminus e; x, y)$.
3. If e is a coloop, $T(\mathcal{M}; x, y) = x T(\mathcal{M}/e; x, y)$.
4. If e is neither a loop nor a coloop, then

$$T(\mathcal{M}; x, y) = T(\mathcal{M} \setminus e; x, y) + T(\mathcal{M}/e; x, y).$$

Thm.

From above example,

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$$T(\mathbb{F}_2^3; 1, x) = 2 + 3x + 2x^2 + x^3$$

$$1. T(M; x, y) = T(M^*; y, x)$$

$$2. T(M; 1, 1) = \# \text{ bases}$$

$$3. T(M; 2, 1) = \# \text{ independent sets}$$

$$4. T(M; 1, 2) = \# \text{ spanning trees}$$

$$5. T(M; 2, 0) = 2^n$$

$$6. T(M; 2, 0) = \# \text{ acyclic orientations}$$

Suppose M is a graphic matroid associated with a graph G .

7. Let $P_G(\lambda) = \#$ colorings of G with λ colors (no two vertices of the same color share an edge). Then

$$P_G(\lambda) = (-1)^{r(G)} \lambda^{K(G)} T(M; 1-\lambda, 0)$$

where $K(G) = \#$ components of G .

8. Suppose G is connected. Let $p \in (0, 1)$. Remove each edge independently with probability p . The probability the resulting graph is connected is

$$p^{|E|-r(M)} (1-p)^{r(M)} T(M; 1, \frac{1}{p}).$$

9. Suppose G is connected, and pick a sink vertex s . The **level** of a recurrent element, c , in the sandpile model (G, s) is

$$\text{level}(c) = \deg(c) - |E| + \deg(s).$$

Then $T(M; 1, y) = \sum r_i y^i$ where $r_i = \#$ recurrents of level i .

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Thm. Let E be a finite set and let $\mathcal{I} \subseteq 2^E$. Then

$M = (E, \mathcal{I})$ is a matroid iff

0) $\emptyset \in \mathcal{I}$.

1) \mathcal{I} is closed under the operation of taking subsets:

$$I \in \mathcal{I} \text{ and } J \subseteq I \Rightarrow J \in \mathcal{I}.$$

2) For each weight function $w: E \rightarrow \mathbb{R}$, the greedy algorithm produces a basis of maximal weight.

HW: ① universal property of Tutte

② graph & dual, Tutte of graph and its dual

③ $0 \leq \text{level}(c) \leq g$ with equality achieved at each end. Is it possible for there to be gaps?