

Definitions of mappings: $\partial: \mathbb{Z}E \rightarrow \mathbb{Z}V_0$ *Note change* $\partial^*: \mathbb{Z}E \rightarrow \mathbb{Z}F_0$
 $e \mapsto e^- - e^+$ $e \mapsto \text{left}(e) - \text{right}(e)$
 $\tilde{\partial} = \partial|_{C^*}$, $\tilde{\partial}^* = \partial^*|_C$

We know the rows and cross-crosses are exact.

Claim: \exists iso. $\Psi: \text{crit}(G) \rightarrow \text{crit}(G^*)$ so that everything commutes. (2)

To define Ψ , let $x \in \text{crit}(G)$. Then $\exists y \in \mathbb{Z}V_0$ s.t. $y \rightarrow x$, (i.e. $\bar{y} = x$), and there exists $z \in \mathbb{Z}E$ s.t. $\partial z = y$. Define

$$\Psi(x) = \overline{\partial^* z}$$

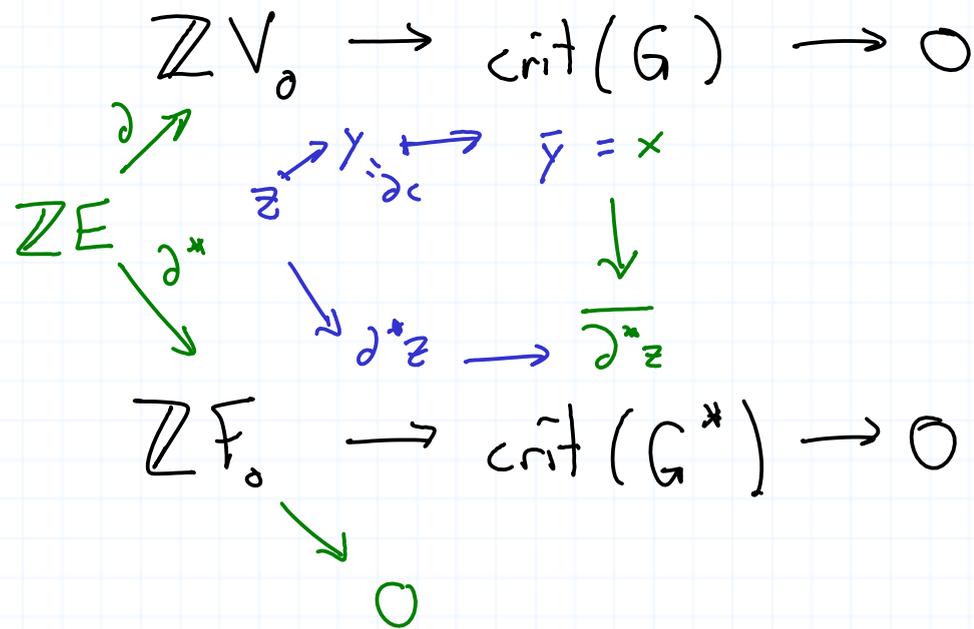
Is Ψ well-defined?

Instead of y , we could choose $y + \tilde{\partial}c^*$ where $c^* \in C^*$,

then pick $w \in \mathbb{Z}E$ with

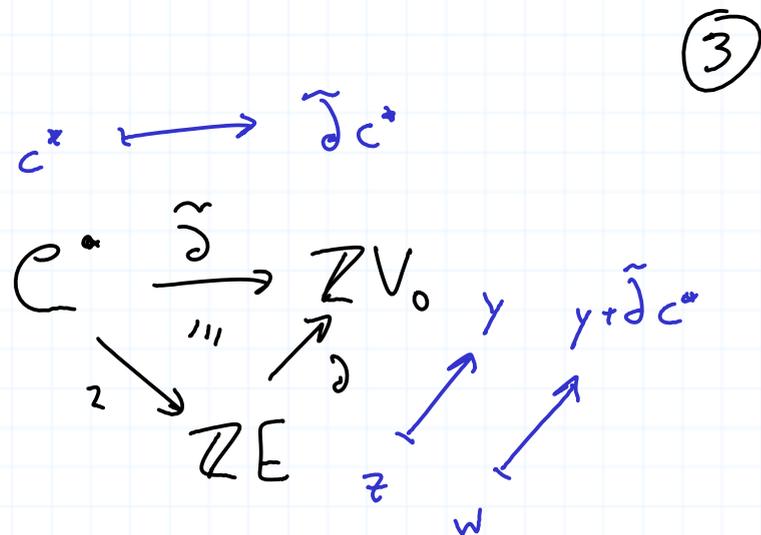
$\partial w = y + \tilde{\partial}c^*$ We must show

$$\overline{\partial^* w} = \overline{\partial^* z}$$



We have that

$$\begin{aligned} \partial(w - z - z(c^*)) &= \partial w - \partial z - \partial z(c^*) \\ &= \gamma + \tilde{\partial}c^* - \gamma - \tilde{\partial}c^* = 0. \end{aligned}$$



Since $\ker \partial = j\mathcal{C}$, $\exists c \in \mathcal{C}$ s.t. $jc = w - z - zc^*$.

Then

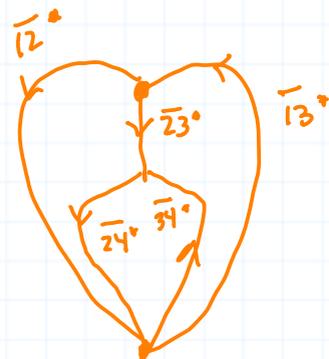
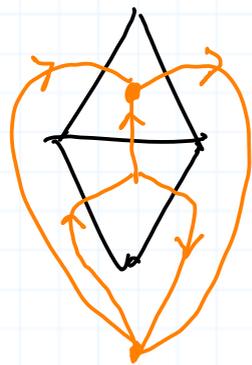
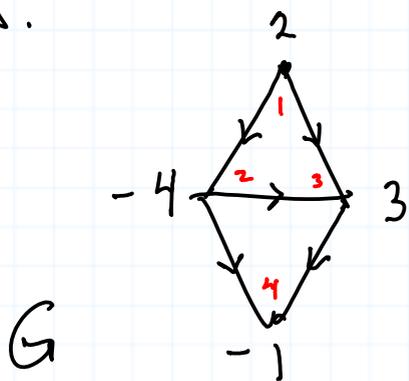
$$\begin{aligned} \tilde{\partial}^*c &= \partial^*jc \\ &= \partial^*w - \partial^*z - \partial^*zc \\ &= \partial^*w - \partial^*z \quad (\text{since } \ker \partial^* = \text{im } z). \end{aligned}$$

Therefore, $\overline{\partial^*w} - \overline{\partial^*z} = \overline{\tilde{\partial}^*c} = 0$ by exactness of the bottom row.

We have shown Ψ is well-defined. By symmetry, we get a well-defined mapping $\mathcal{Q} : \text{crit}(G^*) \rightarrow \text{crit}(G)$.

Given $a \in \text{crit}(G^*)$, pick $z \in \mathbb{Z}E$ with $\partial^* z = a$, and define $\mathcal{Q}(a) = \overline{\partial z}$. It is then clear that \mathcal{Q} and Ψ are inverses.

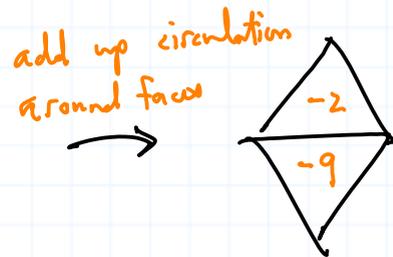
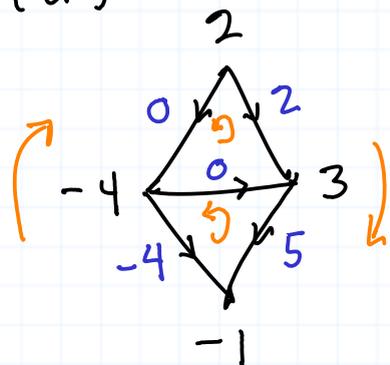
Example



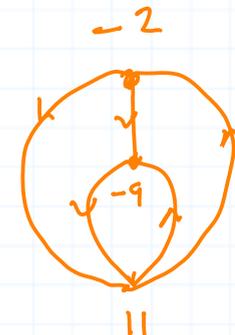
Note: change idea of induced orientation:
 For $e \in E$, let $e^* = (\text{left}(e), \text{right}(e))$
 So $(e^*)^+ = \text{right}(e)$
 $(e^*)^- = \text{left}(e)$

$2v_1 - 4v_2 + 3v_3 - v_4 \in \text{crit}(G)$ $2 \cdot \overline{13} + 5 \cdot \overline{34} - 4 \cdot \overline{24}$ ∂^*

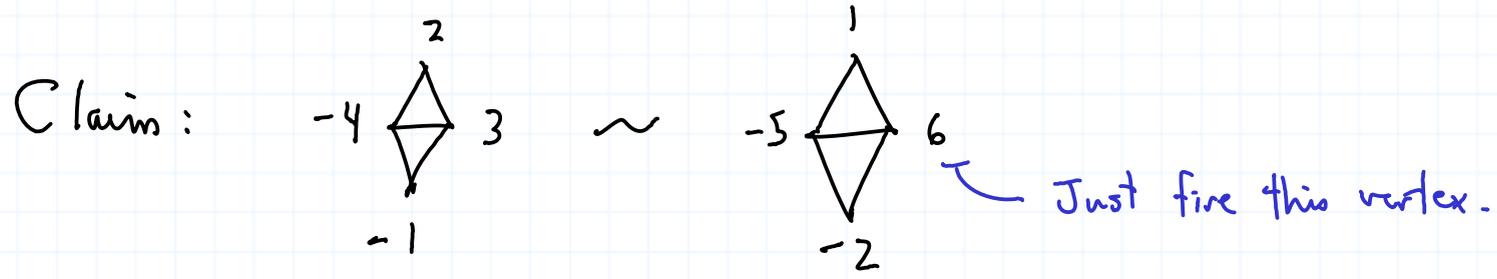
circulation for unbounded face is clockwise



$\textcircled{11}$
 11
 $5 - (-4) - 0 + 2$



Conversely,



Thm Let G be a planar graph. Let W be a collection of edges of G . Then W is a minimal cycle iff W^* , the corresponding edges of the dual graph, is a minimal cut.

Pf/ If W is a cycle, then removal of W from the plane disconnects the plane into two parts: the inside of W and outside (Jordan-Holler thm.)

The edges crossing from inside to outside give the cut W^* . If W contains the edges of a cycle and some more edges, then W^* contains the edges of a cut plus some more edges. If W does not contain a cycle, then it is a tree and hence contains no faces of G .

(6)

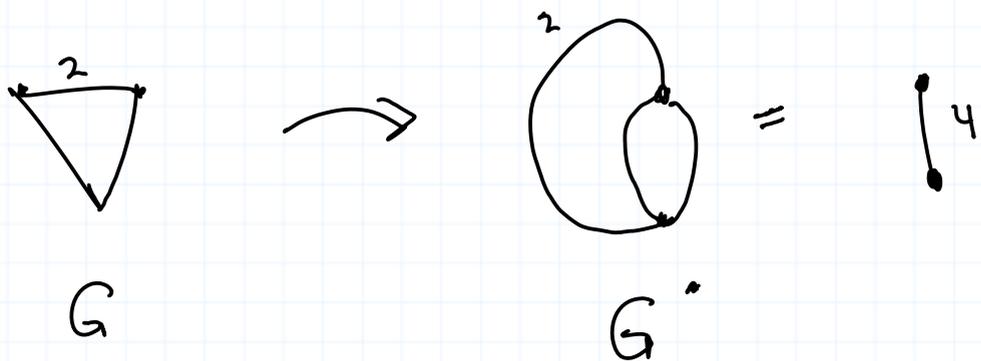
That means there is always a path from each bounded face of G to the unbounded face. Therefore, W^* is not a cut of G^* . \square

Question: How would one use the theorem to show $C \rightarrow C^*$
 $\Sigma_{a \in E} \mapsto \Sigma_{a \in E^*}$
 is a well-defined isomorphism?

Question: Is it always true that $\mathbb{Z}E = C \oplus C^*$? Recall
 For an arbitrary unweighted graph G \leftarrow We've shown $C = (C^*)^\perp$.

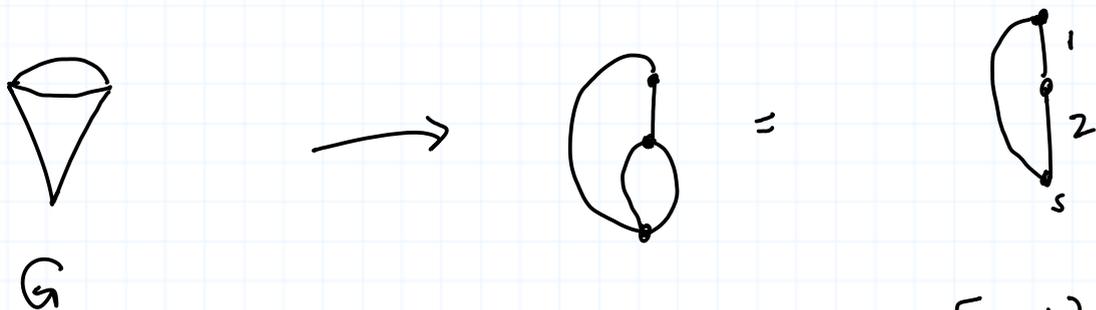
Example from David Krueger

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$$S(G) \cong \frac{\mathbb{Z}}{5\mathbb{Z}}$$

$$S(G^*) = \frac{\mathbb{Z}}{4\mathbb{Z}}$$



$$\det \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = 5$$

$$S(G^*) = \frac{\mathbb{Z}}{5\mathbb{Z}}$$