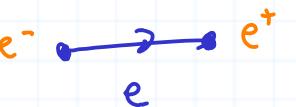


Let  $G = (V, E)$  be an undirected graph, not necessarily connected.

Choose an orientation for each edge. Then, each  $e \in E$  has a head,  $e^+$ , and a tail,  $e^-$ .

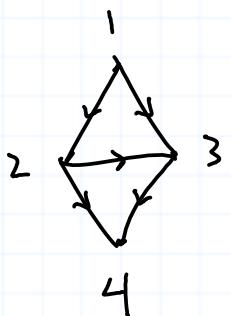


The boundary of  $e \in E$  is  $\partial e = e^+ - e^-$ . Extend  $\partial$  linearly gives

$$\partial: \mathbb{Z}E \rightarrow \mathbb{Z}V$$

An element of  $\ker \partial$  is called a circulation.

Example



$$\partial = \frac{1}{2} \begin{bmatrix} 12 & 13 & 23 & 24 & 34 \\ -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Note ① As a matrix,  $\partial$  is given by the oriented incidence matrix.

② Ordering the vertices and edges, we can identify  $\mathbb{Z}^V$  with

(2)

$\mathbb{Z}^V$  and  $\mathbb{Z}^E$  with  $\mathbb{Z}^E$ . Then, taking the transpose of  $\mathcal{D}$  gives  $\mathcal{D}^t: \mathbb{Z}^V \rightarrow \mathbb{Z}^E$ , and  $\mathcal{D}^t \mathcal{D} = \Delta_G$ .

### CYCLES

Def. A **directed path** in  $G$  is an ordered list of vertices

$P = (v_1, v_2, \dots, v_k)$  where  $(v_i, v_{i+1}) \in E \quad \forall i$ . It is **simple** if no vertex is repeated. It is a **directed cycle** if no vertex is repeated except that  $v_1 = v_k$ . For each  $e \in E$ , define the **sign** of  $e$  in a simple directed path or directed cycle,  $P$ , (with respect to the fixed orientation), is defined by

$$\sigma(P, e) = \begin{cases} 1 & \text{if } e = (v_i, v_{i+1}) \text{ for some } i \text{ and } e^- = v_i \\ -1 & \text{if } e = (v_i, v_{i+1}) \text{ for some } i \text{ and } e^+ = v_i \\ 0 & \text{otherwise ( } e \text{ does not occur in } P\text{).} \end{cases}$$

We can then identify  $P$  with  $\sum_{e \in E} \sigma(P, e) e \in \mathbb{Z}E$ . (3)

Define the (integral) cycle space,  $C \subseteq \mathbb{Z}E$ , as the  $\mathbb{Z}$ -span of all directed cycles.

Cuts An oriented cut of  $G$  is an ordered partition of the vertices into two parts  $(U, U^c)$ . The cut-set corresponding to  $U \subseteq V$ , denoted  $C_U^*$  is the collection of edges with one vertex in  $U$  and the other in  $U^c$ . For each  $e \in E$ , define the sign of  $e$  in  $C_U^*$  by

$$\sigma(C_U^*, e) = \begin{cases} 1 & \text{if } e^- \in U \text{ and } e^+ \in U^c \\ -1 & \text{if } e^+ \in U \text{ and } e^- \in U^c \\ 0 & \text{otherwise.} \end{cases}$$

We can then identify  $C_U^*$  with  $\sum_{e \in E} \sigma(C_U^*, e) e \in \mathbb{Z}E$ . The  $\mathbb{Z}$ -span of the cut-sets is called the cut space of  $G$ , denoted  $C^*$ .

If  $U = \{v\}$ , we write  $c_v^*$  for  $c_U^*$ .

Bases for the cycle and cut spaces

A maximal forest in  $G$  is a union of spanning trees for each connected component of  $G$ . Fix a maximal forest  $F$ . Thus,  $F$  has no cycles and every vertex is on some edge of  $F$ . Let  $\bar{F} = E - F$ . For each edge  $e \in \bar{F}$ , it follows that  $F \cup \{e\}$  has a unique cycle,  $c_e$  such that  $\sigma(c_e, e) = +1$ . We'll see that  $\{c_e : e \in \bar{F}\}$  forms a basis for the cut space. Alternatively, for each  $e \in \bar{F}$ , removal of  $e$  from  $F$  disconnects the tree which is a part of  $F$  in the component of  $G$  containing  $e$ . Partition the vertices of  $G$  into ① those vertices in one of the parts of the disconnected tree and and ② the rest of the vertices of  $G$ . This defines a unique

cut set  $c_e^*$  such that  $\sigma(c_e^*, e) = +1$ . We will see that

(5)

$\{c_e^*: e \in F\}$  forms a basis for the cut space of  $G$ .

Thm.

1.  $C = \ker \delta$
  2. Let  $F$  be a maximal forest of  $G$ . Then  $\{c_e: e \in \bar{F}\}$  is a basis for  $C$  and  $\{c_e^*: e \in F\}$  is a basis for  $C^*$ .
  3.  $\dim_{\mathbb{Z}} C = \#(E) - \#(V) + k$ ,  $\dim_{\mathbb{Z}} C^* = \#(V) - k$ , where  $k$  is the number of components of  $G$ .
  4.  $C = (C^*)^\perp = \{f \in \mathbb{Z}^E: \langle f, g \rangle = 0 \quad \forall g \in C^*\}$  where  $\langle \cdot, \cdot \rangle$  is defined for  $e, e' \in E$  by
- $$\langle e, e' \rangle = \delta(e, e') = \begin{cases} 1 & \text{if } e = e' \\ 0 & \text{if } e \neq e' \end{cases}$$

and extended linearly to all of  $\mathbb{Z}E$ .

5. If  $G$  is connected, then

$$0 \rightarrow C \rightarrow \mathbb{Z}E \xrightarrow{\partial} \mathbb{Z}V \xrightarrow{\text{deg}} \mathbb{Z} \rightarrow 0$$

is exact.

Pf) It is clear that  $C \subseteq \ker \partial$ . For the opposite inclusion, let

$f = \sum a_e e \in \mathbb{Z}E$  and define

$$g = f - \sum_{e \in F} a_e C_e.$$

Then  $\partial g = \partial f$  and  $\text{supp}(g) \subseteq F$ . If  $\text{supp}(g) \neq \emptyset$ , let  $v$  be a vertex of degree 1 in the subgraph determined by  $\text{supp}(g)$

(which must exist since  $\text{supp}(g) \subseteq F$ ). Then  $v \in \text{supp}(\partial g)$ , hence

$\partial f = \partial g \neq 0$ . So if  $f \in \text{ker } \partial$ , we see that  $g=0$ , i.e. ⑦

$$f = \sum_{e \in F} \alpha_e c_e \in C,$$

For part 2, we have just seen that  $\{c_e : e \in \bar{F}\}$  spans  $C = \text{ker } \partial$ .  
These elements are linearly independent since  $\text{supp}(c_e) \cap \bar{F} = \{e\}$ .