

(Γ, s) sandpile model

$I = \mathcal{I}(\Gamma) = \langle x^u - x^v : u-v \in \tilde{L} \rangle$, toppling ideal

Say $\tilde{V} = \{v_1, \dots, v_n\}$ where $i > j$ if v_i is further from the sink

than v_j . Let $R = \mathbb{C}[x_1, \dots, x_n]$ with toppling order $>$.

We have already seen that if c and c' are configurations and $\sigma \geq 0$, $\sigma \neq 0$ is a firing script with $c \xrightarrow{\sigma} c'$, then $x^c > x^{c'}$.

Notation For a script $\sigma \in \mathbb{Z}\tilde{V}$, let

$$\mathcal{T}(\sigma) = x^{(\tilde{\Delta}\sigma)^+} - x^{(\tilde{\Delta}\sigma)^-} \in I.$$

Thm. Let b be any burning configuration, and let σ_b be its script. Then the following is a Gröbner basis for I :

$$\mathcal{M} = \{ T(\sigma) : 0 \leq \sigma \leq \sigma_b \}.$$

Pf/ By definition of I , we have $T(\sigma) \in I \quad \forall \sigma$. Also, we have seen that I is generated by $T(v_1), \dots, T(v_n), T(\sigma_b) = x^{b-1}$.

Hence, $I = \langle T : T \in \mathcal{M} \rangle$. It remains to show that the S -polynomials for the elements of \mathcal{M} reduce to zero upon division by elements of \mathcal{M} .

Let $0 \leq \sigma_i \leq \sigma_b$, $c_i^+ = (\Delta \sigma_i)^+$, and $c_i^- = (\Delta \sigma_i)^-$ for $i = 1, 2$.

Then c_i^- is the configuration obtained from c_i^+ by firing σ_i ($c_i^+ + c_i^- = \Delta \sigma \Rightarrow c_i^+ - \Delta \sigma = c_i^-$). Thus, $T(\sigma_i) = x^{c_i^+} - x^{c_i^-}$ has

leading term $x^{c_i^+}$. Define

$$x^{a_i} = \frac{\text{lcm}(x^{c_1^+}, x^{c_2^+})}{x^{c_i^+}}$$

so that $a_1 + c_1^+ = a_2 + c_2^+ = c$ for some configuration c . We must show that the following reduces to 0 upon division by \mathcal{H} :

$$s = x^{a_1} \tilde{T}(\sigma_1) - x^{a_2} \tilde{T}(\sigma_2) = x^{a_2 + c_2^-} - x^{a_1 + c_1^-}.$$

Since σ_1 and σ_2 are both legal firing scripts from c , so is $\sigma = \max(\sigma_1, \sigma_2)$, defined by $\sigma_v = \max(\sigma_{1,v}, \sigma_{2,v}) \forall v \in \tilde{V}$. Say $c \xrightarrow{\sigma} c'$. Then

$$c = a_i + c_i^+ \xrightarrow{\sigma_i^-} a_i^+ + c_i^- \xrightarrow{\sigma - \sigma_i^-} c'$$

for $i=1, 2$. We have

③

$$x^{a_i^+ + c_i^-} - m_i T(\sigma - \sigma_i) = x^{c'}$$

for some monomial m_i , for $i=1,2$. Hence, s reduces to zero upon division by \mathcal{G} . \square

Thm. There is a unique superstable configuration in each equivalence class of $\mathbb{Z}\tilde{V}$ modulo \tilde{L} . It is the smallest element with respect to toppling order.

Pf/ Two configurations c_1, c_2 are equivalent modulo \tilde{L} iff x^{c_1} and x^{c_2} are equivalent modulo \mathbb{I} . To see this, identify $S(P, s)$ with \mathbb{Z}^n / \tilde{L} and let

$$\psi: R = \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{C}[S(P, s)]$$

$$x_i \mapsto t_{v_i}.$$

We then have $\ker \psi = I$ so that $R/I \xrightarrow{\cong} \mathbb{C}[S(\Gamma, s)]$. ⑤

Therefore, $x^{c_1} = x^{c_2} \pmod I$ iff $t_{c_1} = t_{c_2}$ in $\mathbb{C}[S(\Gamma, s)]$, i.e.,
iff $c_1 = c_2$ in $S(\Gamma, s)$, i.e., iff $c_1 = c_2 \pmod{\tilde{L}}$.

Given any configuration c , reducing x^c via the Gröbner basis

\tilde{L} just described reduces x^c to $x^{c'}$ where $c = c' \pmod{\tilde{L}}$ and c' is superstable. (If $\sigma \geq 0$ is a legal script for c , dividing c by $T(\sigma)$ results in the configuration obtained from c by performing σ as many times as is legal.) Uniqueness of the superstabilization now follows from the uniqueness of the remainder upon division by a

Gröbner basis. \square