Math 412

$$1 \overset{?}{\in} (x^2+y-3,\ xy^2+2x,\ y^3)$$

Yes! $$1 = -\tfrac{1}{27}(y^2+3y+9)(x^2+y-3) - \tfrac{1}{108}(xy^4+3xy^3+7xy^2-6xy-18x)(xy^2+2x)$$
$$+ \tfrac{1}{108}(x^2y^3+3x^2y^2+9x^2y+4)y^3.$$

Let $R = \mathbb{C}[x_1,\dots,x_n]$, and fix a monomial ordering $>$ on R

Def. Let $I \subseteq R$ be an ideal. A **Gröbner basis** for I with respect to $>$ is a subset

$$\{g_1,\dots,g_s\} \subseteq I$$

such that

$$\langle \mathrm{in}(g_1),\dots,\mathrm{in}(g_s)\rangle = \mathrm{in}\, I := \langle \mathrm{in}(f) : f \in I\rangle.$$

Prop. If $J \subseteq I$ are ideals of R, then

$$\mathrm{in}(J) = \mathrm{in}(I) \Rightarrow J = I.$$

Idea of pf/ If $J \subsetneq I$, take an element $f \in I \setminus J$ with smallest initial term. Choose $g \in J$ s.t. $\mathrm{in}(g) = \mathrm{in}(f)$. Now consider $f - g$ to get a contradiction. $\square$

Cor. If $g_1, \dots, g_s$ is a GB for I, the $\langle g_1, \dots, g_s \rangle = I$.

Prop. If $g_1, \dots, g_s$ is a GB for I and $f \in R$, then $f \in I$ iff the remainder of f upon division by $g_1, \dots, g_s$ is 0.

Pf/ Easy. For instance, if $f \in I$, so is the remainder. What is the remainder's initial term? $\square$

Prop. If $g_1, \dots, g_s$ is a GB for I then the collection of monomials not divisible by any $\mathrm{in}(g_i)$ forms a $\mathbb{C}$-basis for R/I.

Pf/ Macaulay's theorem. $\square$

Note: If $f \in R$, then the remainder upon division by $g_1, \dots, g_s$ is the expression for $f \in R/I$ as a linear combination of the basis elements just described.

Buchberger Algorithm

Input $g_1, \dots, g_s$ generating I

Output GB for I

Initialization: Let $C = \{(i,j) : 1 \le i < j \le s\}$, $\mathcal{H} = \{g_1, \dots, g_s\}$.

1. If $C = \emptyset$, halt. Otherwise, pick (i,j) and delete it.

2 Let $m_{ij} = \dfrac{in(g_i)}{\gcd(in(g_i), in(g_j))}$, $m_{ji} = \dfrac{in(g_j)}{\gcd(in(g_i), in(g_j))}$,

and consider the S-polynomial $s_{ij} := m_{ji} g_i - m_{ij} g_j$.

Let h_{ij} be the remainder of s_{ij} upon division by $g_1, \dots, g_s$.

3 If $h_{ij} = 0$, go to step 1. Otherwise

3.1 Set $g_{s+1} = h_{ij}$ and add g_{s+1} to $\mathcal{G}$

3.2 Add $(i, s+1)$ for $1 \le i \le s$ to C

3.2 Replace s by $s+1$.

3.3. Go to step 1.

$\square$

Example $I = \langle x^2, xy + y^2 \rangle$, with deglex monomial ordering.

$C = \{(1,2)\}$, $\mathcal{G} = \{ \underset{g_1}{x^2}, \underset{g_2}{xy + y^2} \}$

Choose $(1,2)$. Then $s_{12} = y\underset{g_1}{x^2} - x(\underset{g_2}{xy+y^2}) = -xy^2$. Reduce s_{12} using $\mathcal{G}$: $-xy^2 - y(xy-y^2) = y^3 = h_{12}$.

New $\mathcal{G} = \{x^2, xy+y^2, y^3\}$ and now $C = \{(1,3),(2,3)\}$.

Choose $(1,3)$: $s_{13} = y^3(x^2) - x^2(y^3) = 0$. So $h_{13} = 0$.

Choose $(2,3)$: $s_{23} = y^2(xy+y^2) - x(y^3) = y^4$. Reduce s_{23} using $\mathcal{G}$: $y^4 + y(y^3) = 0 = h_{23}$.

Done: a GB for I is $\{x^2, xy+y^2, y^3\}$.

$\mathbb{C}$-basis for R/I: $1, x, y, \cancel{x^2}, \cancel{xy}, y^2, \cancel{x^3}, \cancel{x^2y}, \cancel{xy^2}, \cancel{y^3}, \cancel{\cdots}$

$\{1, x, y, y^2\}$