Math 412: Sandpiles

8/30/10

www.reed.edu/~davidp/412

www.reed.ed/~davidp/sand

Good first reference: "Chip-firing and rotor-routing on directed graphs."

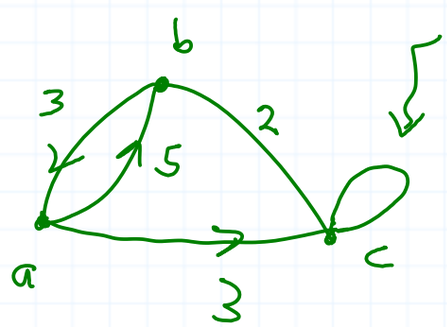
Structure of class: weekly HW + projects

Handouts: "What is a sandpile?"
"Chip-firing and ..."

Notation and First Definitions

- $\Gamma = (V, E)$ directed multigraph with vertex set V and edge multiset E. Both V and E are assumed finite
- For $u, v \in V$, define the weight $wt(u,v) :=$ # edges from u to v

Example:



loops allowed

5 edges from a to b

3 edges from b to a

3 edges from a to c

2 edges from b to c and from c to b

$wt(c,a) = 0$, $wt(c,c) = 1$

- For $e = (u,v) \in E$, the tail of e is $e^- = u$, and the head of e is $e^+ = v$

- For $v \in V$,

$$\text{outdeg}(v) = \sum_{w \in V} wt(v,w) = \sum_{e : e^- = v} 1 = \text{out degree of } v$$

$$\text{indeg}(v) = \sum_{u \in V} wt(u,v) = \sum_{e : e^+ = v} 1 = \text{in degree of } v$$

- A vertex $v \in V$ is (globally) accessible if there is a directed path from each vertex of Γ to v.

- A sandpile graph is a pair (Γ, s) where Γ is a directed multigraph and s is an accessible vertex of Γ. In this case s is called the sink of Γ.

- $\mathbb{N}V = \{ \sum_{v \in V} n_v v : n_v \in \mathbb{N} \}$ = free commutative monoid on V

  $\mathbb{N} = \{0, 1, 2, \ldots\}$

  $\mathbb{Z}V = \{ \sum_{v \in V} n_v v : n_v \in \mathbb{Z} \}$ = free abelian group on V

  $\tilde{V} = V \setminus \{s\}$ = non-sink vertices

  $\mathbb{N}\tilde{V}$ = free commutative monoid on $\tilde{V}$

  $\mathbb{Z}\tilde{V}$ = free abelian group on $\tilde{V}$

- A configuration (of sand) on (Γ, s) is an element

$$c = \sum_{v \in \tilde{V}} c_v v \in \mathbb{N}\tilde{V}.$$

If $v \in \tilde{V}$, then $c_v < \text{outdegree}(v)$ the c is stable at v; otherwise c is unstable at v. The configuration c is stable if it is stable at all $v \in \tilde{V}$.

We think of a configuration c as a pile of sand on Γ.

If c is unstable at v, the we can fire (topple) c at v to get a new configuration $\tilde{c}$ defined by

$$\tilde{c} = c - \mathrm{outdeg}(v)\, v + \sum_{e : e^- = v,\, e^+ \neq s} e^+$$

$$= c - \mathrm{outdeg}(v)\, v + \sum_{w \in \tilde{V}} \mathrm{wt}(v,w)\, w$$

functions $\phi : V \to \mathbb{Z}$

- The Laplacian of Γ is the operator $L : \mathbb{Z}^V \to \mathbb{Z}^V$ defined for $\phi \in \mathbb{Z}^V$ by

$$L\phi : V \to \mathbb{Z}$$

$$v \mapsto \sum_{e : e^- = v} \big(\phi(v) - \phi(e^+)\big) = \mathrm{outdeg}(v)\,\phi(v) - \sum_{w \in V} \mathrm{wt}(v,w)\,\phi(w).$$

The standard basis for $\mathbb{Z}^V$ is $\{v^*\}_{v \in V}$ where

$$v^*(w) = \delta(v,w) = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}.$$

Fixing an ordering $v_1, \ldots, v_n$ of the vertices of Γ gives an isomorphism

$$\begin{aligned} \mathbb{Z}^V &\xrightarrow{\cong} \mathbb{Z}^n \\ v_i^* &\longmapsto e_i . \end{aligned}$$

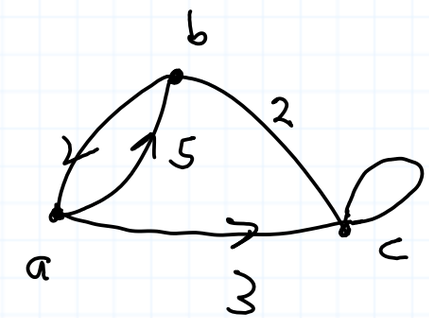
With respect to this isomorphism, the Laplacian becomes an $n \times n$ matrix, also called L, where

$$L_{ij} = \begin{cases} \operatorname{outdeg}(v_i) - \operatorname{wt}(v_i, v_i) & \text{if } i = j \\ -\operatorname{wt}(v_i, v_j) & \text{otherwise.} \end{cases}$$

Let $D = \mathrm{diag}(\mathrm{outdeg}(v_1), \ldots, \mathrm{outdeg}(v_n))$ be the diagonal matrix of degrees, and let $A = (\mathrm{wt}(v_i, v_j))_{1 \le i,j \le n}$ be the adjacency matrix of Γ. Then,

$$L = D - A.$$

Example The Laplacian of the graph



is

$$\begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 8 & -5 & -3 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \end{array} = \begin{bmatrix} 8 & & \\ & 3 & \\ & & 3 \end{bmatrix} - \begin{bmatrix} 0 & 5 & 3 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$L = D - A$$