

Lattice Ideals

Let A be an abelian group generated by elements a_1, \dots, a_n .

Define $\varphi: \mathbb{Z}^n \rightarrow A$, and let $L = \ker \varphi$. Let $\{t_a: a \in A\}$
 $e_i \mapsto a_i$

be indeterminates and consider the **group algebra**

$$\mathbb{C}[A] = \mathbb{C}[t_a: a \in A]$$

where $t_a t_b = t_{a+b} \quad \forall a, b \in A$. Let $R = \mathbb{C}[x_1, \dots, x_n]$ be the

polynomial ring. **Monomial notation:** For $c \in \mathbb{N}^n$, define $x^c = \prod_{i=1}^n x_i^{c_i}$ where

x_1, \dots, x_n are indeterminates. Define

$$\psi: R \rightarrow \mathbb{C}[A]$$

$$x_i \mapsto t_{a_i}.$$

Thus, for $c \in \mathbb{N}^n$, $\Psi(x^c) = \prod_{i=1}^n t_{a_i}^{c_i} = t_{\sum c_i a_i} = t_{c \cdot a}$.

(2)

Def. The **lattice ideal** for $L = \ker \Phi$ is $I(L) = \ker \Psi$.

Hence, $R/I(L) \cong \mathbb{C}[A]$.

For any $u \in \mathbb{Z}^n$, define $u_i^+ = \max\{0, u_i\}$, $u_i^- = \max\{0, -u_i\}$ so that $u = u^+ - u^-$ and $\text{supp}(u^+) \cap \text{supp}(u^-) = \emptyset$.

Thm. \star (1) $I(L) = \text{Span}_{\mathbb{C}} \{x^u - x^v : u - v \in L\}$. (The vector space span forms an ideal.)

(2) Let l_1, \dots, l_k be \mathbb{Z} -module generators for $L \subseteq \mathbb{Z}^n$. Then $I(L)$ is the saturation of the ideal $J = \langle x^{l_i^+} - x^{l_i^-} : i=1, \dots, k \rangle$ with respect to $\prod_{i=1}^n x_i$:

$$I(L) = \left\{ f \in R : \left(\prod_{i=1}^n x_i \right)^m f \in J \text{ for some } m \geq 0 \right\}.$$

Note: Since $R/\mathcal{I}(\mathcal{T}) \xrightarrow{\psi} \mathbb{C}[A]$ is an isomorphism of rings, it's also an isomorphism of \mathbb{C} -vector spaces. If $\exists U \subseteq \mathbb{N}^n$ s.t.

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$\mathcal{B} = \{x^u : u \in U\}$ is a basis for $R/\mathcal{I}(\mathcal{T})$, then

$$\{\psi(x^u) : u \in U\} = \{t_{\sum u_i a_i} : u \in U\} = \{t_a : a \in A\},$$

Where the second equality holds since $\{t_a : a \in A\}$ is a basis for $\mathbb{C}[A]$.

Thus, ψ induces a bijection between \mathcal{B} and A , which induces a group structure on \mathcal{B} . A standard way to find \mathcal{B} is to pick a **monomial ordering** on R and let \mathcal{B} be a **normal basis** with respect to that ordering. We'll define these terms later.

Toppling Ideals

(4)

Def. Let (Γ, s) be a sandpile graph, and let $S(\Gamma, s)$ be its sandpile group. The (inhomogeneous) sandpile ideal is the lattice ideal for

$$\tilde{L} = \text{image}(\tilde{\Delta}): \quad \mathcal{I}(\Gamma) = \text{Span}_{\mathbb{C}} \{ x^u - x^v : u - v \in \tilde{L} \}.$$

(Order the vertices v_1, \dots, v_n to identify $\mathbb{Z}\tilde{V}$ with \mathbb{Z}^n .) For each vertex v_i

define the **toppling polynomial**, $T_i = x_i^{\text{outdeg } v_i - \text{wt}(v_i, v_i)} - \prod_{j \neq i} x_j^{\text{wt}(v_i, v_j)}$.

Lemma Let c be a configuration on Γ and suppose $c \rightarrow \tilde{c}$ via a sequence of vertex firings, then $x^c = x^{\tilde{c}}$ modulo the ideal generated by the T_i .

Pf/ It suffices to consider the case where $c \rightarrow \tilde{c}$ by firing a single vertex v_i . Then, $x^{c - (\text{outdeg}(v_i) - \text{wt}(v_i, v_i))e_i} T_i = x^c - x^{\tilde{c}}$, where $e_i = i^{\text{th}}$ std. basis vector. \square

Thm. Let b be a burning configuration. Then

$$I(\Gamma) = \langle T_i : 1 \leq i \leq n \rangle + \langle x^b - 1 \rangle.$$

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Pf/ Let J denote the right-hand side. We have $J \subseteq I(\Gamma)$

since $T_i = x^{l_i^+} - x^{l_i^-}$ where l_i is the v_i -th column of $\bar{\Delta}$. Also,

$b = b - \vec{0} \in L$, hence, $x^b - x^{\vec{0}} = x^b - 1 \in I(\Gamma)$. By Thm. \star ,

we know $I(\Gamma)$ is the saturation of J with respect to $\prod_{i=1}^n x_i$.

So it suffices to show J is already saturated. Let $f \in R$

and suppose $(\prod_{i=1}^n x_i)^m f \in J$. Since $x^b - 1 \in J$, we have

$(x^b - 1)f = 0 \pmod{J}$, i.e. $x^b f = f \pmod{J}$. Hence, $x^{kb} f = f \forall k \geq 0$.

Since b is a burning configuration, for $k \gg 0$ we have

$kb \rightarrow (m, \dots, m) + c$ for some configuration c . Hence, by the lemma,

$f = x^{kb} f = (\prod_{i=1}^n x_i)^m x^c f = 0 \pmod{J}$, (since $(\prod_{i=1}^n x_i)^m f \in J$). \square