

(1)

Math 412

Thm. The following are equivalent for $D \in \text{div}(\Gamma)$:

- 1) $D \in \mathcal{N}$
- 2) $K - D \in \mathcal{N}$
- 3) \exists max'l superstable c s.t. $D \sim c - s$ [3.5) \exists min'l recurrent \tilde{c} s.t. $D_{\max} - D \sim \tilde{c} + \deg(s)s$]
- 4) $D_{\max} - D$ is minimally alive
- 5) $|D| = \emptyset$ and $|D + v| \neq \emptyset \quad \forall v \in V.$

Pf/ 1 \Rightarrow 4) $D \in \mathcal{N} \Rightarrow |D| = \emptyset$. Suppose $E \sim D$, then $|D| = \emptyset \Rightarrow E \neq 0$

$\Rightarrow \exists v$ s.t. $E_v < 0 \Rightarrow \exists v$ s.t. $(D_{\max} - D)_v > 0$. Thus, $D_{\max} - D$ is alive. To show $D_{\max} - D$, calculate its degree:

$$\deg(D_{\max} - D) = \deg D_{\max} - (g-1) = (\deg \sum_v (\deg v - 1)) - (g-1)$$

$$= (|E_P| - |V_P|) - (|E_P| - |V_P|) = |E_P|.$$

(2)

(4 \Rightarrow 2) Suppose $E \sim K - D = D_{\max} - \vec{I} - D$. Since $D_{\max} - D$ is minimally alive and $E + \vec{I} \sim D_{\max} - D$, we've seen $\text{supp}(E + \vec{I}) \neq V$. So $\exists v \in V$ s.t. $(E + \vec{I})_v = 0$. Hence, $E_v = -1$ and E is not effective. Hence, $|K - D| = \emptyset$. To check the degree: $D_{\max} - D$ minimally alive $\Rightarrow \deg(D_{\max} - D) = |E_P|$. So $\deg(K - D) = \deg(D_{\max} - D - \vec{I}) = |E_P| - |V_P| = g-1$.

(2 \Rightarrow 1) The implications $1 \Rightarrow 4 \Rightarrow 2$ gives $D \in \mathcal{N} \Rightarrow K - D \in \mathcal{N}$.

Replacing D by $K - D$ says $K - D \in \mathcal{N} \Rightarrow K - (K - D) = D \in \mathcal{N}$.

(4 \Rightarrow 5) $D_{\max} - D$ minimally alive $\Rightarrow D \in \mathcal{N}$, as we've already seen. So $|D| = \emptyset$.

We also know that for all $v \in V$, $D_{\max} - D - v$ is not alive.

(3)

So \exists stable $E \sim D_{\max} - D - v$. Hence, $D + v \sim D_{\max} - E \geq 0$, which shows $|D + v| \neq \emptyset$.

(5 \Rightarrow 4) Suppose $|D| = \emptyset$ and $E \sim D_{\max} - D$. Then $D_{\max} - E \sim D \Rightarrow D_{\max} - E \neq 0 \Rightarrow \exists v \in V$ s.t. $E_v \geq \deg v$. Thus, $D_{\max} - D$ is alive.

Further, $\forall v \in V$ we have $|D + v| \neq \emptyset$. So $\exists E \sim D + v$ with $E \geq 0$.

Hence, $D_{\max} - D - v \sim D_{\max} - E$ and $D_{\max} - E$ is stable. Hence, $D_{\max} - D - v$ is not alive.

(3 \Leftrightarrow 3.5) \Leftarrow superstable $\Leftrightarrow c_{\max} - c$ recurrent . \square

Riemann - Roch

(4)

We've seen RR is equivalent to the following:

RR1) $\forall D \in \text{div}(\Gamma), \exists v \in N$ s.t. exactly one of $|D|$ and $|v-D|$ is nonempty.

RR2) If $\deg D = g-1$, then $|D|$ and $|K-D|$ are either both empty or both nonempty.

Pf / (RR1) Let $D \in \text{div}(\Gamma)$. If $|D| \neq \emptyset$, take any $v \in N$. Then $|v-D| = \emptyset$ since $|v| = \emptyset$. (Take $E \in |D|$. Then $F \in |v-D| = |v-E| \Rightarrow F \sim v - E \Rightarrow F + E \in |v|$.)

Now suppose $|D| = \emptyset$. Write $D \sim c - ks$ with c superstable for (Γ, s) .

Choose a maximal superstable $\tilde{c} \geq c$, and define $v = \tilde{c} - s$. Then $v \in N$ and $v - D \sim (\tilde{c} - s) - (c - ks) \geq 0$. So $|v - D| \neq \emptyset$.

(RR2) We've seen $D \in N \Leftrightarrow K-D \in N$. Given that D and $K-D$ both have degree $g-1$, this is equivalent to saying $|D| = \emptyset \Leftrightarrow |K-D| = \emptyset$. Therefore, RR2 holds. For RR1, if $|D| = \emptyset$, then $D_{\max} - D$ is strict.