

(P, S) Eulerian graph. If $S = v_1, \dots, v_n$ is an ordering of V_P , we've defined the minimal effective divisor D^S relative to S .

It has the following properties:

- 1) $D^S \leq E$ for any divisor E that is effective relative to S .
- 2) Defining $D_0 = D_n = D$, then $D_i = D_{i-1} - \Delta v_i$ for $i \geq 1$, we have
 - a) $D_n \xrightarrow{v_1} D_1 \xrightarrow{v_2} \dots \xrightarrow{v_{n-1}} D_{n-1}$
 - b) $v_i \notin \text{supp}(D_i)$

- c) D_i is the minimal effective divisor for $v_{i+1}, v_{i+2}, \dots, v_n, v_1, \dots, v_i$.

Thm. Let $S = v_1, \dots, v_n$ be an ordered list of the vertices, and let $E \in |D^S|$. Then $\text{supp}(E) \neq V$.

(2)

Pf) Given $E \in |D^S|$, let σ be the minimal effective script for $E \xrightarrow{\sigma} D$. Let $l = \max \{i \in \{1, \dots, n\} : \sigma_{V_i} = 0\}$.

By minimality of σ , we know l exists. Then let $\tau = \sigma + \sum_{i=1}^l v_i$.

Thus, $\text{supp}(\tau) = V$, so $\tilde{\tau} := \tau - \vec{v}$ is effective and firing $\tilde{\tau}$ from E has the same effect as firing τ . Note that $E \xrightarrow{\tau} D_l$ where D_l is the result of firing v_1, \dots, v_l from D . We noted above that $v_l \notin \text{supp}(D_l)$. Now $E \xrightarrow{\tilde{\tau}} D_l$, and by construction, $\tilde{\tau}_{v_l} = 0$.

This means that $E_{v_l} = 0$. So $v_l \notin \text{supp}(E)$. \square

Def. A divisor D is **alive** if D is unstable (i.e., $\exists v \in V$ s.t. $D_v \geq \deg(v)$) and $\forall D' \sim D$, we have D' is also unstable.

The divisor is a **minimal alive divisor** if D is alive but for all $v \in V$, $D - v$ is not alive.

(3)

Prop. The following are equivalent for $D \in \text{div}(\Gamma)$.

- 1) D is alive.
- 2) Fixing any $s \in V$ as a sink, \exists a recurrent c on (Γ, s) and an integer $k \geq \deg(s)$ such that $D \sim c + ks$.
- 3) \exists an ordering of the vertices S and an $E \sim D$ such that E is effective relative to S .

Pf / (1 \Rightarrow 2) Suppose D is alive. \wedge stabilize w.r.t. s to get $D \sim c_1 + k_1 s$ for some stable configuration c_1 on (Γ, s) . Since D is alive $c_1 + k_1 s$ is not stable. So $k_1 \geq \deg(s)$. Fire s , and stabilize again to get $c_2 + k_2 s$. Repeat to form c_1, c_2, c_3, \dots . Now, since

$$\begin{aligned} * D - D_s &= D - \Delta(\tilde{\Gamma} - \sum_{v \neq s} s) = D + \sum_{v \neq s} \Delta v \\ &= D + b + ks \end{aligned}$$

$$s \begin{bmatrix} \tilde{\Delta} & * & * \\ * & \ddots & * \\ * & * & \ddots & * \\ \vdots & & & 1 \\ 0 & & & \vdots \\ 0 & & & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ i \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ * \end{bmatrix}.$$

Γ is Eulerian, firing the sink adds the minimal burning configuration, b , to c_i . Thus, $c_i = (c + i b)^\circ$. We've seen that for $i \gg 0$, we have $(ib)^\circ = \varepsilon$, the identity of $S(\Gamma, s)$, a recurrent element. Hence, $c_i = c_{i+1} \dots$ for $i \gg 0$. At that point, after firing the sink, each non-sink vertex fires in the subsequent stabilization.

($2 \Rightarrow 3$) If $D \sim c + ks$ with c recurrent and $k \geq \deg s$, then firing s is legal and adds the minimal burning configuration to c . Since c is recurrent, we can stabilize c relative to s by firing all of the non-sink vertices in some order. Thus, we can let $E := c + ks$.

($3 \Rightarrow 1$) Suppose $\exists E \sim D$ effective relative to the ordering v_1, \dots, v_n of V . Given $F \sim D$, let $E \rightarrow F$ via the minimal effective script σ . Take the minimal k such that $\sigma_{v_k} = 0$. Then

$$F_{v_k} = (E - \Delta\sigma)_{v_k} \geq (E - \Delta v_1 - \dots - \Delta v_{k-1})_{v_k} \geq \deg(v_k), \quad (5)$$

so v_k is unstable in F . \square

A slight refinement of the above argument gives

Prop. The following are equivalent for $D \in \text{div}(\Gamma)$.

- 1) D is minimally alive
- 2) Fixing any $s \in V$ as a sink, \exists a minimal recurrent c on (Γ, s)
and an integer $k \geq \deg(s)$ such that $D \sim c + ks$. \leftarrow This integer k
must be $\deg(s)$.
- 3) \exists an ordering of the vertices S and an $E \sim D$ such that
 $E = D^S$, i.e. E is the minimal effective divisor relative to S .

If c' is recurrent
and $c' \neq c$, then
 $c' \neq c$.

(6)

Cor. If D is minimally alive, then $\deg D = |E_P|$.

Pf/ From above, \exists an ordering of the vertices $S: v_1, \dots, v_n$ s.t.

$D^S \sim D$. Then $\deg D = \deg D^S = |E_P|$. \square

Cor. If c is a minimal recurrent configuration (i.e. c is recurrent and there does not exist $v \in \tilde{V}$ s.t. $c - v$ is transient) then

$$\deg c = |E| - \deg(s).$$

Pf/ From above, $c + \deg(s)s \sim D^S$ for some ordering of the vertices, S .

Hence, $\deg(c) + \deg(s) = \deg D^S = |E_P|$. \square

Note: We've shown that the minimal currents (resp., maximal superstable) all have the same degree. (This does not hold for directed graphs, in general!)