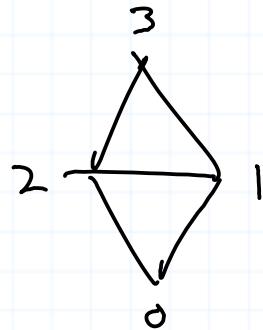


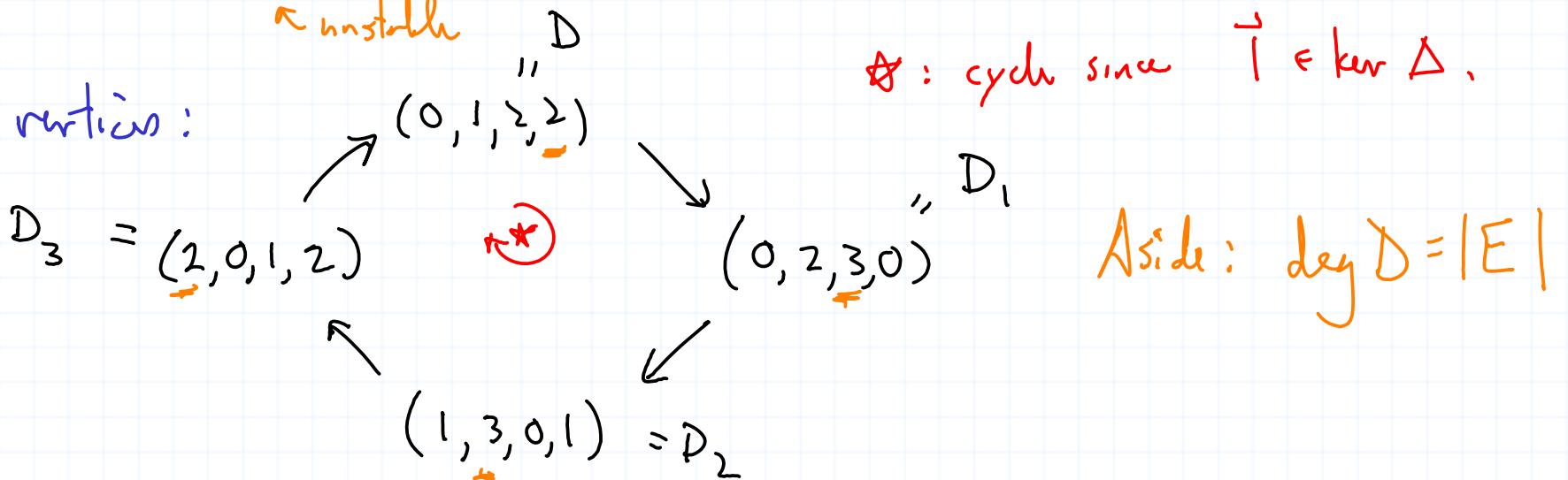
Find an effective divisor of least degree on
so that the vertices 3, 2, 1, 0 can fire,
in that order, maintaining an effective divisor.



Solution: $D = (0, 1, 2, 2)$

$\overline{\text{unstable}}$

Firing unstable vertices:

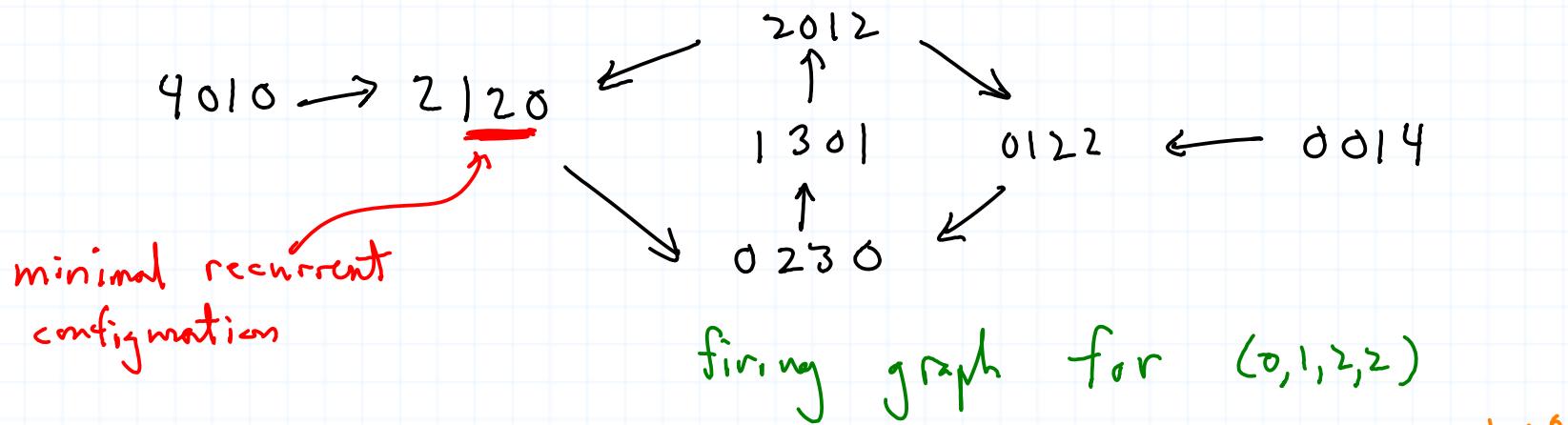


Facts we will soon see:

- * Let D_{\max} be the "maximal stable divisor": $(1, 2, 2, 1)$.

Then $D_{\max} - (0, 1, 2, 2) = (1, 1, 0, -1)$. This turns out to be a non special divisor on Γ . (Note: $\deg(1, 1, 0, -1) = 1 = g - 1$.) (0)

* $|I(0, 1, 2, 2)| = 2012, 4010, 0122, 0014, 2120, 1301, 0230$



$|I(0, 1, 2, 2)|$ contains a unique element of the form $c + k \cdot 0$ where c is a minimal recurrent element of $(\Gamma, 0)$, namely, $(2, 1, 2, 0)$. Note that $(2, 1, 2, 0)$ is "minimally alive" with respect to the vertex ordering $0, 2, 1, 3$.

(Γ, s) a sandpile graph (soon to be Eulerian)

A **firing script** $\sigma \in \mathbb{Z}^V$ is **legal** for $D \in \text{div}(\Gamma)$ when

$\forall v \in V$, if $D_v \geq 0$, then $(D - \Delta\sigma)_v \geq 0$. So no vertices acquire negative values as a result of firing σ . A sequence of firing scripts $\sigma_1, \sigma_2, \dots$ is legal from D if σ_1 is legal from D , then σ_2 is legal from $D - \Delta\sigma_1$, i.e., after firing σ_1 , and so on.

The divisor D is **effective relative to σ** if D is effective and σ is legal from D . Given a sequence S of scripts $\sigma_1, \sigma_2, \dots$, D is effective relative to S if D is effective and $\sigma_1, \sigma_2, \dots$ is a legal firing sequence from D . The divisor D is a **minimal effective divisor** relative to $\sigma_1, \sigma_2, \dots$ if D is effective relative to $\sigma_1, \sigma_2, \dots$.

(2)

and given any divisor E effective relative to $\sigma_1, \sigma_2, \dots$, with $E \neq D$
we have $E \notin D$.

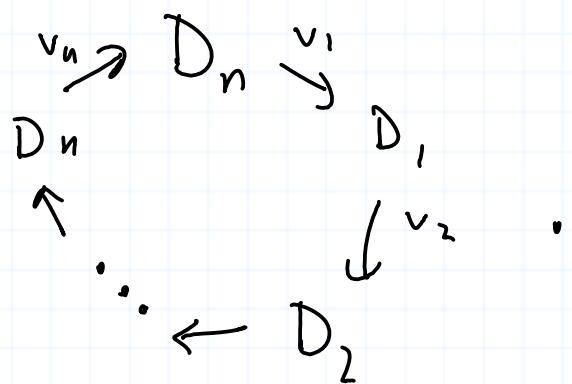
Prop. Given any sequence of scripts $S = \sigma_1, \sigma_2, \dots$, $\exists!$ minimal
effective divisor D^S relative to S . Further, after firing $\sigma_1, \dots, \sigma_i$
from D^S one obtains the minimal effective divisor relative to $\sigma_{i+1}, \sigma_{i+2}, \dots$

Pf/ See Wilmes'. The idea is to add just enough sand so that
each script is legal. \square

In the special case of firing distinct vertices $S = v_1, \dots, v_n$, the
minimal effective divisor relative to S is given by

$$D_{v_i}^S \stackrel{\star}{=} \sum_{i < j \leq k} \text{wt}(v_i, v_j).$$

Specializing further, if v_1, \dots, v_n is a (non-redundant) list of all the vertices, then after firing v_1, \dots, v_n , the vector \vec{t} has been fired. Since $\vec{t} \in \ker \Delta$, we are back at the original divisor. Letting $D_0 = D_n = D$, we have



Assume P
is Eulerian so
 $\vec{t} \in \ker \Delta$.

Note: Suppose v_1, \dots, v_n are all the vertices, in some order, and D is the corresponding minimal effective divisor, and the D_i are as above. Then

- 1) $\deg D = |E_P|$ (by \star) ← (Careful: This might only hold for Eulerian graphs)
- 2) $v_i \notin \text{supp}(D_i)$. ← (check!)

(4)

Now **assume** Γ is Eulerian. If $D \sim D'$, there is a minimal effective firing script, σ , such that

- 1) $D \xrightarrow{\sigma} D'$, i.e. $D' = D - \Delta\sigma$
- 2) $\sigma \geq 0$
- 3) Given any $\tau \geq 0$ s.t. $D \xrightarrow{\tau} D'$, we have $\tau \geq \sigma$.

To form σ , take any τ s.t. $D \xrightarrow{\tau} D'$. Since $\text{ker } \Delta = \vec{1}$, we can add just the right multiple of $\vec{1}$ to τ to get σ .

Note: $\exists v \in V$ s.t. $\sigma_v = 0$, by construction. (Here is one place we need the Eulerian assumption.)

Thm. Let $S = v_1, \dots, v_n$ be an ordered list of the vertices, and let $E \in |D^S|$. Then $\text{supp}(E) \neq V$.

PF/ Next time.