

Let (Γ, s) be a sandpile model w/ Γ a directed multigraph.

Def. A **(firing) script** is an element $\sigma \in \mathbb{Z}^{\tilde{V}}$. If c is a configuration on Γ , **firing** the script σ yields the configuration $c - \tilde{\Delta}\sigma$. We write

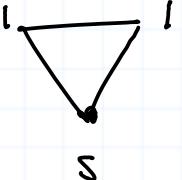
$$c \xrightarrow{\sigma} c - \tilde{\Delta}\sigma.$$

The script firing σ is **legal** from c if $c - \tilde{\Delta}\sigma$ is a configuration, i.e. $c - \tilde{\Delta}\sigma \geq 0$.

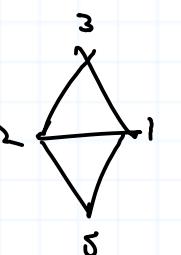
If σ is the characteristic function for a subset of \tilde{V} , $c \xrightarrow{\sigma} c - \tilde{\Delta}\sigma$ is called a **vertex firing**.

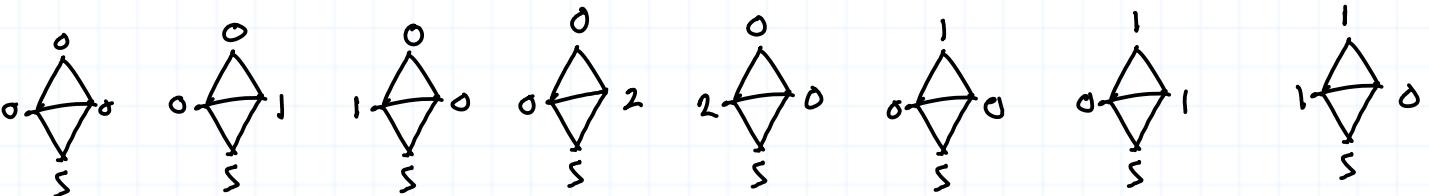
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Def. A configuration is **superstable** if it has no legal script firings $\sigma \geq 0$.

Example  is stable but not superstable. Firing both

nonsink vertices gives the superstable configuration .

Example The superstables for  are



Def. (Toppling order) An ordering of the vertices $V = (v_1, \dots, v_{n+1})$ is a **toppling order** if $i < j$ whenever v_i is further from the sink than v_j .

Note: To create a toppling order, first choose v_1 to be any vertex with maximal distance from the sink. Then choose v_2 to be any vertex with maximal distance from the sink, not counting v_1 , etc.

Finally, v_{n+1} is the sink, itself.

Def. (Graded reverse lexicographic ordering $\xleftarrow{\text{grlex}}$ on $\mathbb{Z}\tilde{V}$ or $\mathbb{Z}V$.)

Fix a toppling order on the vertices $V = (v_1, \dots, v_{n+1})$.

Let $D, D' \in \mathbb{Z}V = \text{div}(\Gamma)$. We say $D \geq_{\text{gr}} D'$ if $\deg D > \deg D'$ or if $\deg D = \deg D'$ and $D(v_i) < D'(v_i)$ for the maximal i s.t. $D(v_i) \neq D'(v_i)$. This \geq_{gr} is called the **toppling order** for $\mathbb{Z}V$. Then define the toppling order \geq_{gr} in $\mathbb{Z}\tilde{V}$ by restriction: if $a, b \in \mathbb{Z}\tilde{V}$, say $a \geq_{\text{gr}} b$ if $a \geq_{\text{gr}} b$ as elements of $\mathbb{Z}V$.

Remark: Roughly, $D \geq_{gr} D'$ for $D, D' \in \text{div}(\Gamma)$ means either
 $\text{deg } D > \text{deg } D'$ or D has less stuff on some vertex close the sink.

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Prop. $\mathbb{Z}\widehat{V}$ and $\mathbb{Z}V$ are totally ordered groups under \geq_{gr} .

Pf/ It is straightforward to check, $\forall a, b, c \in \mathbb{Z}\widehat{V}$ (resp., $a, b, c \in \mathbb{Z}V$)

1) Exactly one of the following holds $a >_{gr} b$, $a = b$, $b >_{gr} a$.

2) $a >_{gr} b$ and $b >_{gr} c \Rightarrow a >_{gr} c$.

3) $a >_{gr} b \Rightarrow a + c >_{gr} b + c$.

□

Prop. Fix a tipping ordering $>_{jr}$ on V . For all $\sigma \in \mathbb{Z}\widehat{V}$ with $\tau \geq 0$, we have $\widehat{\Delta} \sigma \geq_{gr} \vec{0}$ with equality iff $\sigma = \vec{0}$.

Pf/ It suffices to show that $\widehat{\Delta}_v \geq_{gr} \vec{0} \quad \forall v \in \widehat{V}$. If v

is adjacent to the sink, then $\deg \tilde{\Delta}v > 0 = \deg \vec{0}$, hence, $\tilde{\Delta}v \succ_{gr} \vec{0}$. (5)

Otherwise, we have $\tilde{\Delta}v \succ_{gr} \vec{0}$ since v has an out-edge to a vertex that is closer to the sink.

Cor. Let c be a configuration of sand on Γ .

If $\sigma \geq 0$, $\sigma \neq 0$, and $c \xrightarrow{\sigma} c'$, then $c \succ_{gr} c'$.

Pf/ If $\sigma \geq 0$, $\sigma \neq 0$, then $\tilde{\Delta}\sigma \succ_{gr} 0$, hence; $c \succ_{gr} c - \tilde{\Delta}\sigma = c'$. □

Prop. Fix \succ_{gr} as above. Every non-empty subset of $N\tilde{V}$ has a unique smallest element under \succ_{gr} .

Pf/ Let $C \subseteq N\tilde{V}$ be nonempty. We may assume that all the elements of C has the same degree. In that case, C is finite, and the result follows since \succ_{gr} is a total ordering. □

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Thm. Each equivalence class of \mathbb{Z}^V modulo \tilde{I} contains a unique superstable element. It is the smallest configuration in the equivalence class with respect to \geq_{fr} .

Pf/ The only proof I know at the moment is essentially a result about the Gröbner basis of the "toppling ideal".

A proof of the result for Eulerian graphs appears in "chip firing + Rotor-routing" (and for undirected graphs in Baker and Norine).

Fact. (Also from Gröbner basis theory) Let b be any burning configuration, and let σ_b be its script. A configuration c is superstable iff it has no legal firing scripts $\sigma \leq \sigma \leq \sigma_b$.

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Cor. If Γ is Eulerian, a configuration is superstable iff it has no legal vertex firings.

Pf/ If Γ is Eulerian, $\vec{\tau}$ is a burning script. \square

Thm. The recurrent element in each equivalence class modulo \tilde{L} is the largest stable configuration in the equivalence class with respect to \geq_{gr} .

Pf/ Let c be the largest stable configuration in an equivalence class, and let b be a burning configuration with script σ_b .

Then $c+b \sim c$, and since $(c+b) - \tilde{A}\sigma_b = c \geq 0$, we have $c+b \geq_{gr} c$. So $c+b$ is unstable at some vertex v .

It must be that $v \in \text{supp}(\sigma_b)$. Let $\sigma_i = \sigma_b - v$ and define

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$c' = c + b - \tilde{\Delta}v$. Then $c' \sim c$ and since $b - \tilde{\Delta}v = \tilde{\Delta}\sigma_i$, with $\sigma_i \geq 0$, we have $c' \geq_{gr} c$, with equality iff $\sigma_i = 0$. If $c' \neq c$, then c' is unstable. Repeating the argument inductively, we see that $(c + b)^\circ = c$, hence c is recurrent. \square

Cor. A configuration c is recurrent iff $c_{\max} - c$ is superstable.

Pf/ Immediate from the two previous theorems. \square