Thm. The formula $r(D) - r(K-D) \overset{(RR)}{=} \deg D + 1 - g$ holds for all divisors iff the following two properties hold:

(RR1) $\forall D \in \mathrm{div}(\Gamma)$, $\exists v \in \mathcal{N}$ s.t. exactly one of $|D|$ and $|v-D|$ is nonempty.

(RR2) $\forall D \in \mathrm{div}^{g-1}(\Gamma)$, we have $|D|$ and $|K-D|$ are either both empty or both nonempty.

Last time: (RR2) $\Leftrightarrow$ $r(K) \geq g-1$.

Pf. of thm. / ($\Rightarrow$) Suppose (RR) holds. If $\deg D = g-1$, then (RR) says $r(D) = r(K-D)$. Thus, (RR2) follows. We now show (RR1) follows. If $v \in \mathcal{N}$, then

lemma from last time

$$-1 = r(\nu) = r(D + \nu - D) \geq r(D) + r(\nu - D).$$

Therefore, $|D| \neq \emptyset \Rightarrow r(D) \geq 0 \Rightarrow r(\nu - D) = -1 \Rightarrow |\nu - D| = \emptyset$ for all $\nu \in \mathcal{N}$. So the real point is to show that (RR) implies that if $|D| = \emptyset$, $\exists\, \nu \in \mathcal{N}$ s.t. $|\nu - D| \neq \emptyset$.

Suppose $\deg D = d$ and $|D| = \emptyset$. We would like to find $E \geq 0$ with $E \sim \nu - D$ for some $\nu \in \mathcal{N}$, i.e. $E \geq 0$ such that $\deg(E + D) = g - 1$ and $r(E + D) = -1$. Suppose there is no such E. Therefore, for all $E \geq 0$ with $\deg E = g - 1 - d$ we have $r(E + D) \geq 0$. By (RR), we get

$$r(K - D - E) = r(E + D) + \deg(E + D) + 1 - g = r(E + D) \geq 0.$$

It follows that $r(K-D) \geq g-1-d$, whence, by (RR),

$$r(D) = r(K-D) + \deg D + 1 - g \geq g-1-d+d+1-g = 0.$$

So $|D| \neq \emptyset$, a contradiction.

($\Leftarrow$) Now suppose (RR1) and (RR2) hold.

<u>Lemma</u> (RR1) implies

$$r(D) = \min_{\substack{D' \sim D \\ v \in N}} \left\{ \deg^+(D' - v) \right\} - 1$$

where $\deg^+\left(\sum_{v \in V} n_v \cdot v\right) = \sum_{\substack{v \in V: \\ n_v \geq 0}} n_v$.

Pf/ Later. $\square$

Back to proof of thm.: Let $D' \sim D$ and let $v \in N$. By (RR2),

there is a bijection $\mathcal{N} \longrightarrow \mathcal{N}$, $v \longmapsto K - v =: \bar{v}$

$\left(\deg(K-v) = \deg K - \deg v = g-1 \right.$ and $\left. |K-v| = \emptyset \text{ since } |v| = \emptyset \right)$

Then $v - D' = K - D' - \bar{v}$, so

$$\star \quad \deg^+(D'-v) - \deg^+((K-D')-\bar{v}) = \deg^+(D'-v) - \deg^+(v-D')$$

$$= \deg(D'-v)$$

$$= \deg(D-v) = \deg D + 1 - g.$$

Now fix D^* and v^* such that $\deg^+(D^* - v^*) = \min\limits_{\substack{D' \sim D \\ v \in \mathcal{N}}} \{\deg^+(D'-v)\} = r(D)$

Then $\deg^+((K-D')-\bar{v}) = \deg^+(D'-v) - (\deg D + 1 - g) \geq \deg^+(D^* - v^*) - (\deg D + 1 - g)$

for all $K-D'$ and $\bar{v}$ says

$$= r(D) - (\deg D + 1 - g)$$

$$r(K-D) = \min_{\substack{K-D' \sim K-D \\ \bar{v} \in N}} \{ \deg^+(K-D'-\bar{v}) \} \geq \deg^+(D^* - v^*) - (\deg D + 1 - g).$$

$$= r(D) - (\deg D + 1 - g).$$

But letting $D' = D^*$ and $\bar{v} = \overline{v^*} = K - v^*$, we see by ☆ that the inequality is actually an equality. $\square$