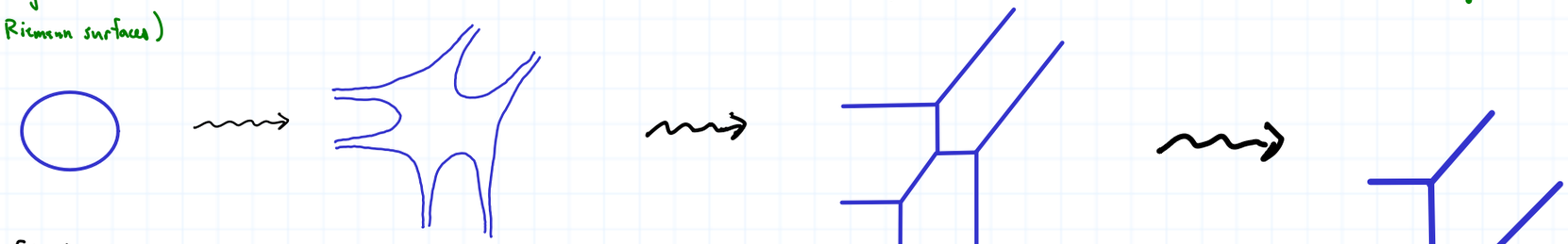
Tropical Geometry

Reference: Baker + Norrine

Algebraic curves (= Riemann surfaces) $\rightsquigarrow$ Amoebas $\rightsquigarrow$ Tropical curves $\rightsquigarrow$ Graphs



$f(x,y)=0$

$\{(\log|x|, \log|y|) : f(x,y)=0\}$

metrized graphs

(Γ, s) sandpile graph

Defs. A divisor on Γ is an element of $\mathbb{Z}V$. The collection of all divisors is denoted $\mathrm{Div}(\Gamma) = \mathbb{Z}V$. For $D = \sum n_v v \in \mathrm{Div}(\Gamma)$, the degree of Γ is $\deg D := \sum n_v$. The collection of divisors of

degree k is denoted $\mathrm{Div}^k(\Gamma)$. Let $\mathcal{L} = \mathrm{image}(\Delta : \mathbb{Z}V \to \mathbb{Z}V)$ $\mathrm{Div}(\Gamma) \to \mathrm{Div}(\Gamma)$

The elements of $\mathcal{L}$ are principal divisors. The class group of Γ is $Cl(\Gamma) := \mathbb{Z}V/\mathcal{L}$ = divisors modulo principal divisors.

Divisors D and D' are linearly equivalent, denoted $D \sim D'$, if their classes are equal in the class group: $[D] = [D']$ in $Cl(\Gamma)$.

Thus, $D \sim D'$ if D' can be obtained from D by firing and reverse firing vertices (including the sink possibly).

A divisor E is effective if $E \geq 0$, i.e. $E_v \geq 0 \ \forall v \in V$.

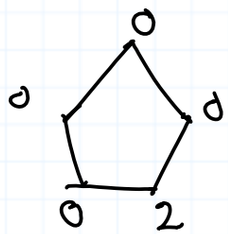
The complete linear system of a divisor D is

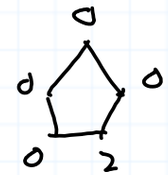
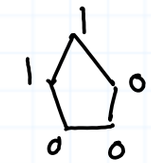
$$|D| = \{ E \in \mathrm{div}\,\Gamma : E \sim D \text{ and } E \geq 0 \}$$

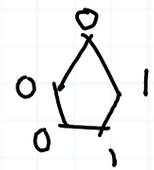
The dimension of the linear system $|D|$ is

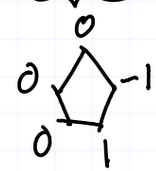
$$r(D) = \begin{cases} -1 & \text{if } |D| = \emptyset \\ \max\{d \in \mathbb{Z} : |D-E| \neq \emptyset \ \forall E \in \mathrm{div}^d(\Gamma),\ E \geq 0\} & \text{if } |D| \neq \emptyset. \end{cases}$$

Example $\Gamma = C_5$, the cycle graph with 5 vertices

$D =$ 

$|D| = \{$  , ,  $\}$

$r(D) = 1$: Letting $E =$  , $|D - E| = \emptyset$



If $F \in |D-E|$, then $F \geq 0$ and $\deg F = 0 \Rightarrow F = 0$. This would mean

$$\begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for some $a_1, \dots, a_5 \in \mathbb{Z}$. Solving this equation over $\mathbb{Q}$ gives $\vec{a} = (\frac{1}{5}, -\frac{3}{5}, -\frac{2}{5}, -\frac{1}{5}, 0) + s\vec{1}$ for $s \in \mathbb{Q}$.

No integer solutions.

Assumption Now suppose Γ is undirected (☆ what happens if Γ is Eulerian? Keep an eye on this ☆)

Def. For Γ undirected, define the genus of $\Gamma = (V, E)$ to be $g = |E| - |V| + 1$. Define the canonical divisor to be

$$K = \sum_{v \in V} (\deg(v) - 2)\, v = D_{max} - \vec{1}$$

where D_{max} is the maximal stable divisor: $D_{max,v} = \deg(v) - 1 \quad \forall v \in V$.

Theorem (Riemann-Roch - Baker - Norine) $\forall D \in \operatorname{div}(\Gamma)$

$$r(D) - r(K - D) = \deg D + 1 - g.$$