

Dependence of $S(\Gamma, s)$ on s

①

Γ any graph, $\Delta: \mathbb{Z}V \rightarrow \mathbb{Z}V$ Laplacian.

Let $\mathbb{Z}V_0 = \{ c = \sum_v n_v v \in \mathbb{Z}V : \sum_v n_v = 0 \}$. Since $\text{image}(\Delta) \subseteq \mathbb{Z}V_0$, let $\Delta_0: \mathbb{Z}V \rightarrow \mathbb{Z}V_0$ be defined by $\Delta_0(v) = \Delta(v)$ $\forall v \in V$. We also use Δ_0 to denote $\Delta_0: \mathbb{Z}V/\ker \Delta \rightarrow \mathbb{Z}V_0$.

Def. The **critical group** for Γ is

$$\text{critical}(\Gamma) = \text{cok } (\Delta_0).$$

Thm. 1. Let $s \in V$ be an accessible vertex, $\tilde{V} = V - \{s\}$, and let $\tilde{\Delta}$ be the reduced Laplacian for (Γ, s) . For each $v \in V$, let τ_v be the sum of the weights of all spanning trees directed into v , let $\tilde{\tau}_v = \tau_v / \text{gcd}_v(\tau_v)$, and let $\tilde{\tau} = \sum \tilde{\tau}_v v$, a generator for $\ker \Delta$.

(1) There is a commutative diagram with exact rows

(2)

$$\begin{array}{ccccccc}
 0 & \rightarrow & \mathbb{Z}\tilde{V} & \xrightarrow{\tilde{\Delta}} & \mathbb{Z}\tilde{V} & \rightarrow & S(\Gamma, s) \rightarrow 0 \\
 & & \downarrow z & & \downarrow \varepsilon & & \downarrow \\
 0 & \rightarrow & \mathbb{Z}V & \xrightarrow{\Delta_0} & \mathbb{Z}V_0 & \rightarrow & \text{critical}(\Gamma) \rightarrow 0
 \end{array}$$

~~$\mathbb{Z}V$~~ $\xrightarrow{\text{ker } \Delta}$

where $\varepsilon(v) = v - s \quad \forall v \in V$ and $z(v) = v \in \mathbb{Z}V / \text{ker } \Delta \quad \forall v$.

(2) There is a short exact sequence

$$0 \rightarrow \mathbb{Z}/\mathbb{Z}_{t_s} \xrightarrow{\alpha} S(\Gamma, s) \xrightarrow{\beta} \text{critical}(\Gamma) \rightarrow 0$$

← really, the recurrent equivalent to this.

where $\alpha(1) = - \sum_{v \in \tilde{V}} \text{wt}(s, v)v$ and $\beta(c) = c - \deg(c)s$.

[If $c = \sum_{v \in \tilde{V}} n_v v$, then $\deg c := \sum n_v$.]

Pf/ Exactness of the rows is immediate. For commutativity,

$$\varepsilon \tilde{\Delta} v = \varepsilon \left((\text{outdeg } v) \cdot v - \sum_{w \in \tilde{V}} \text{wt}(v, w) w \right)$$

$$= \text{outdeg}(v) - \sum_{w \in V} \text{wt}(v, w) w \quad (\text{summing over } V \text{ rather than } \tilde{V}).$$

$$= \Delta_0 v = \Delta_0 \tilde{z} v.$$

The mapping $\mathcal{S}(\Gamma, s) \rightarrow \text{critical}(\Gamma)$ is induced by ε .

For part (2), apply the snake lemma, using the fact that

$$\varepsilon \text{ is invertible } (\varepsilon^{-1} \left(\sum_{v \in V} n_v v \right) = \sum_{v \in \tilde{V}} n_v v):$$

(4)

$$\begin{array}{ccccccc}
 & \text{ker } z & & & \text{ker } \bar{\varepsilon} & & \\
 & \downarrow & & \downarrow & \downarrow & & \\
 0 \rightarrow \mathbb{Z}\tilde{V} & \xrightarrow{\tilde{\Delta}} & \mathbb{Z}\tilde{V} & \rightarrow S(\Gamma, s) \rightarrow 0 & & & \\
 & \downarrow z & & \downarrow \varepsilon & & \downarrow \bar{\varepsilon} & \\
 0 \rightarrow \mathbb{Z}V & \xrightarrow{\Delta_0} & \mathbb{Z}V_0 & \rightarrow \text{critical}(P) \rightarrow 0 & & & \\
 & \downarrow \text{ker } \Delta & & \downarrow & & & \\
 & & & 0 & & & \text{cok } \bar{\varepsilon} \\
 & \downarrow & & & & & \\
 & \text{cok } z & & & & &
 \end{array}$$

By the snake lemma, there is an exact sequence

$$0 \rightarrow \text{ker } z \rightarrow 0 \rightarrow \text{ker } \bar{\varepsilon} \rightarrow \text{cok } z \rightarrow 0 \rightarrow \text{cok } \bar{\varepsilon} \rightarrow 0,$$

whence $\text{ker } z = 0$, $\text{ker } \bar{\varepsilon} = 0$ and $\text{ker } \bar{\varepsilon} \cong \text{cok } z$.

We know that $\ker \Delta = \bar{\mathbb{Z}}$; hence, $\text{cok } z = \frac{\mathbb{Z}}{\bar{\mathbb{Z}}_s \mathbb{Z}}$. (5) \square

Def. Let Γ be any directed multigraph. An **Eulerian path** in Γ is a directed path that contains each edge exactly once. An **Eulerian circuit** is an Eulerian path that begins and ends at the same vertex. Γ is **Eulerian** if it contains an Eulerian circuit.

Thm. 2. A directed multigraph Γ is Eulerian iff it is strongly connected and $\text{indeg}(v) = \text{outdeg}(v) \quad \forall v \in V$

PF / Exercise. \square

$\forall u, v \in V, \exists$ [↑] directed path
in Γ from u to v .

Cor. If Γ is Eulerian, then $S(\Gamma, s) = \text{critical}(\Gamma) \quad \forall s \in V.$ (6)

Thus, $S(\Gamma, s)$ does not depend on $s.$

Pf/ If Γ is Eulerian, then $\text{indeg}(v) = \text{outdeg}(v) \quad \forall v \in \Gamma(v).$

Thus, $\vec{1} \in \Delta.$ There exists at least one directed spanning tree in $\Gamma.$ [To see this, fix a vertex $s.$ Given a vertex $v \neq s,$ form a directed path from v to s not visiting any vertex twice. Given a third vertex $u,$ form a similar path to $s,$ following v 's path if a vertex from the v 's path is reached. Continue with the rest of the vertices, one by one.] Therefore, by the matrix-tree theorem, Δ

has a non-zero minor of size $|V|-1.$ Therefore, $\ker \Delta = \langle \vec{1} \rangle.$

Using the notation of theorem 1, we have $\vec{1}_v = 1 \quad \forall v \in V,$ and the result follows. \square