

Math 412 HW 8

①

up to linear equivalence

1. a) $\mathcal{C}(\Gamma)$ trivial $\Rightarrow \exists! K \in \text{div}(\Gamma)$ and $\exists! g \in \mathbb{Z}$ s.t.

$$(\text{RR}) \quad r(D) - r(K-D) = \deg D + l-g \quad \forall D \in \text{div}(\Gamma).$$

Pf/ Letting $D=0$, for RR to hold, we need $r(0) - r(K) = l-g$,

i.e., $r(K) = g-l$. Letting $D=K$, we need $r(K) - r(0) = \deg K + l-g$,

i.e., $r(K) = \deg K + l-g$. Hence, $\deg K = 2g-2$.

Since $\mathcal{C}(\Gamma)$ is trivial, by direct calculation,

$$r(D) = \begin{cases} -l & \text{if } \deg D < 0 \\ \deg D & \text{if } \deg D \geq 0. \end{cases}$$

For $\deg D \gg 0$, we have $r(D) - r(K-D) = \deg D + l-g \Rightarrow$

$$\deg D - (-1) = \deg D + 1 - g \Rightarrow g = 0 \Rightarrow \deg K = 2g - 2 = -2. \quad (2)$$

We now check RR works with K any divisor of degree -2

and with $g = 0$:

Cases: $\deg D \leq -2$ $r(D) - r(K-D) = -1 - (-\deg D - 2) = \deg D - 1 = \deg D + 1 - g.$

$\deg D = -1$ $r(D) - r(K-D) = -1 - (-1) = 0 = \deg D + 1 - g.$

$\deg D \geq 0$ $r(D) - r(K-D) = \deg D - (-1) = \deg D + 1 = \deg D + 1 - g. \quad \square$

b) $\text{critical}(\Gamma) = \frac{\mathbb{Z}}{3\mathbb{Z}}$ generated by $A = x-z$. Then $\text{Cl}(\Gamma) \xrightarrow{\cong} \mathbb{Z} \oplus \text{critical}(\Gamma)$
 $D \mapsto (\deg D, D - (\deg D)z)$

says the class group in degree d is generated by
 $dz, A + dz, 2A + dz.$

Calculation shows

(3)

$$r(D) = \begin{cases} -1 & \text{if } \deg D < 0 \text{ or if } \deg D = 0 \text{ and } D \neq 0 \\ 0 & \text{if } D = 0 \\ \deg D - 1 & \text{if } \deg D > 0. \end{cases}$$

As above, we know $r(K) = g-1$ and $\deg K = 2g-2$, if it exists.

For $D \gg 0$, we have

$$\begin{aligned} r(D) - r(K-D) &= \deg D + 1-g \Rightarrow (\deg D - 1) - (-1) = \deg D + 1 - g \\ &\Rightarrow g = 1 \Rightarrow \deg K = 0, r(K) = 0. \end{aligned}$$

Since $\deg K = 0$ and $r(K) = 0$, we need $K = 0$.

Check: $\deg D < 0 \quad r(D) - r(K-D) = -1 - (\deg D - 1) = \deg D = \deg D + 1 - g$.

$D \neq 0$ and $\deg D = 0 \quad r(D) - r(K-D) = -1 - (-1) = 0 = \deg D + 1 - g$.

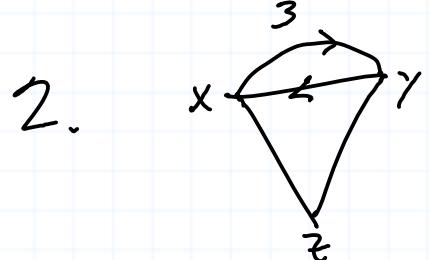
(4)

$$D = 0$$

$$r(D) - r(K-D) = 0 - 0 = \deg D + (-g).$$

$$\deg D > 0$$

$$r(D) - r(K-D) = (\deg D + 1) - (-1) = \deg D = \deg D + 1 - g. \quad \square$$



a) Buring script: $\tilde{\Delta} = \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$, $\tilde{\Delta}[1] = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} -3 \\ 1 \end{bmatrix} + \tilde{\Delta}[1] = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = b, \quad \sigma_b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$GB = \left\{ x^{(\tilde{\Delta}\sigma)^+} - x^{(\tilde{\Delta}\sigma)^-} : 0 < \sigma \leq \sigma_b \right\}.$$

$$\sigma = \begin{bmatrix} 1 \\ 0 \end{bmatrix} : x^4 - y^3 \quad | \quad \sigma = \begin{bmatrix} 0 \\ 2 \end{bmatrix} : y^4 - x^2 \quad | \quad \sigma = \begin{bmatrix} 1 \\ 2 \end{bmatrix} : x^2y - 1$$

$$\sigma = \begin{bmatrix} 0 \\ 1 \end{bmatrix} : y^2 - x \quad | \quad \sigma = \begin{bmatrix} 1 \\ 1 \end{bmatrix} : x^3 - y \quad |$$

$$GB = \left\{ \underline{x^4 - y^3}, \underline{y^2 - x}, \cancel{x^4 - x^2}, \cancel{x^3 - y}, x^2y - 1 \right\}.$$

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2 b) Reduce:

$$x^4 - y^3 \xrightarrow{x^3-y} y^3 - xy \xrightarrow{y^2-x} 0$$

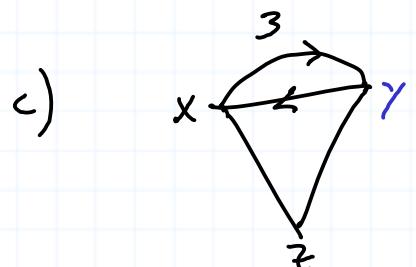
$$\cancel{y^2-x}$$

$$y^4 - x^2 \xrightarrow{y^2-x} xy^2 - x^2 \xrightarrow{y^2-x} 0$$

$$\cancel{x^3-y}$$

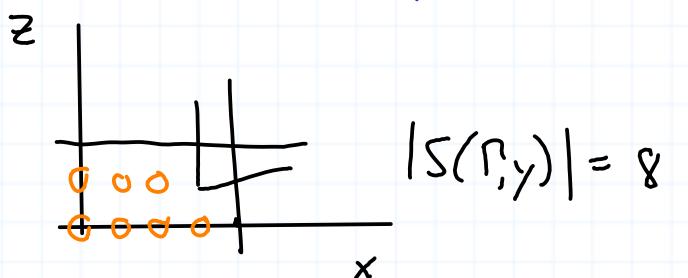
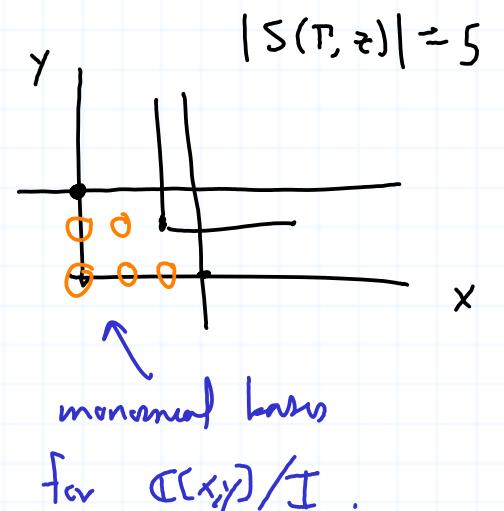
$$\cancel{x^2y} - 1$$

Thus, the reduced GB is $\{y^2 - x, x^3 - y, x^2y - 1\}$

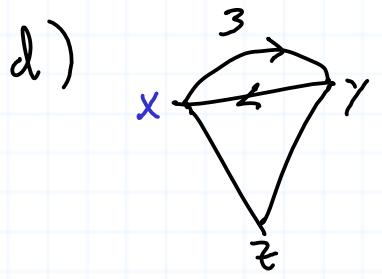


$$\tilde{\Delta} = \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \sigma_{\mathbf{b}} = [1]$$

reduced GB = $\{x^4 - z, z^2 - x, x^3z - 1\}$.

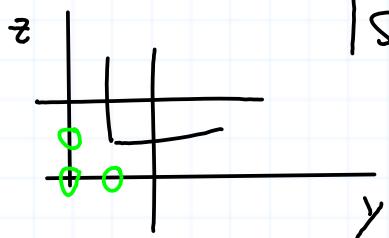


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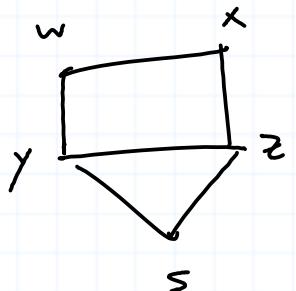
$$\tilde{\Delta} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \sigma_b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

reduced GB = $\{y^2 - z, z^2 - y, yz - 1\}$.



$$|S(r, x)| = 3$$

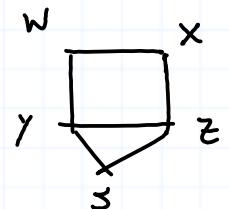
3.



$$\tilde{\Delta} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix}$$

$$\begin{aligned} \text{GB} = \{ & \underset{1000}{w^2 - xy}, \underset{0100}{x^2 - wz}, \underset{0010}{y^3 - wz}, \underset{0001}{z^3 - xy}, \underset{1100}{wx - yz}, \underset{1010}{wy^2 - xz}, \underset{1001}{w^2 z^3 - x^2 y^2} \\ & \underset{0110}{x^2 y^3 - w^2 z^2}, \underset{0101}{x z^2 - wy}, \underset{0011}{y^2 z^2 - wx}, \underset{1100}{x y^2 - z^2}, \underset{1101}{w z^2 - y^2}, \underset{1011}{w y z^2 - x^2}, \\ & \underset{0111}{x y^2 z - w^2}, \underset{1111}{y z - 1} \} \end{aligned}$$

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The basis elements coming from non-well-connected cuts are:

$$* \quad w^2z^3 - x^2y^2 \xrightarrow{w^3-xy} xy\bar{z}^3 - x^2\bar{y}^2 \xrightarrow{z^3-xy} 0$$

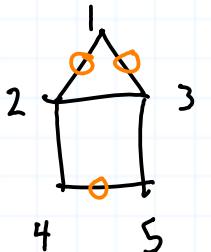
$$* \quad x^2y^3 - w^2z^2 \xrightarrow{x^2-wz} y^3wz - w^3z^2 \xrightarrow{y^3-wz} 0$$

$$* \quad y^2\bar{z}^2 - wx \xrightarrow{yz-1} w\bar{x} - y\bar{z} \xrightarrow{1100} 0$$

$$* \quad wy\bar{z}^2 - x^2 \xrightarrow{wz^2-y^2} y^3 - x^2 \xrightarrow{y^3-wz} x^2 - wz \xrightarrow{0100} 0$$

$$* \quad xy^2\bar{z} - w^2 \xrightarrow{xy^2-z^2} \bar{z}^3 - w^2 \xrightarrow{\bar{z}^3-xy} w^2 - xy \xrightarrow{1000} 0.$$

4 a) A disconnecting set that is not a cut-set:



$$\{12, 13, 15\}$$

Easier: The collection of all edges of $K_3 = \Delta$.

Any cut-set containing these edges would have to separate 1 from 2+3, and would have to separate 4+5. The only possibilities for the cuts would be $(\{1,4\}, \{2,3,5\})$ and $(\{1,5\}, \{2,3,4\})$, both of whose cut-sets contain more edges than just 12, 13, and 15.

(b) A minimal disconnecting subset X divides the graph into 2 connected components.

Let (U, U^c) be the corresponding partition. Let Y be the corresponding cut set. Then every edge of Y is necessary to disconnect U from U^c (since the corresponding induced subgraphs

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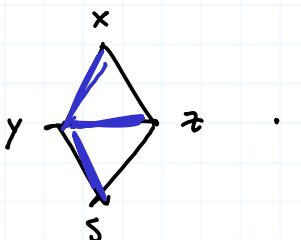
for U and U^c are connected). But X disconnects U from U^c . So $X \supseteq Y$. Minimality then says $X = Y$.

(c) Let F be the cut set corresponding to the cut (U, U^c) .

If $T \setminus F$ has exactly 2 components and $e \in E \setminus F$, then $F \cup \{e\}$ is connected. Hence, F is a minimal cut set. Suppose $T \setminus F$ has at least 3 components. Since T is connected, all the components are connected to U^c by some element of F . Let \tilde{U} be one of these components, and assume, w.o.l.o.g., $\tilde{U} \subsetneq U$. Let H be the cut corresponding to (\tilde{U}, \tilde{U}^c) . If $e \in H$, then e connects different components of $T \setminus F$, hence $e \in F$. So $G \subseteq F$. There is an edge $e \in F$ connecting 2 components of $T \setminus F$, neither of which is \tilde{U} . Hence, $e \notin G$. So $G \subsetneq F$. Thus, F is not a bond.

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d) Fix a tree T in Γ . We have seen that $\{\hat{c_e} : e \in T\}$ spans the cut space. It is clear that each of these $\hat{c_e}$ is a bond. Hence, the bonds span the cut space. They do not form a basis however. Consider



$$\begin{aligned}\{ys, zs\} \\ \{xz, ys, zs\} \\ \{xy, xz\}\end{aligned}$$

with the tree indicated in blue. The corresponding basis consists of bonds, as we have seen, but does not include the bond $\{xz, ys, zs\}$.

5. G oriented, unweighted.

(1)

a) Suppose G is connected and show

$$0 \rightarrow C \rightarrow \mathbb{Z}E \xrightarrow{\partial} \mathbb{Z}V \xrightarrow{\deg} \mathbb{Z} \rightarrow 0$$

is exact.

Pf/ We've already shown $C = \ker \partial$, and \deg is clearly surjective

(fix $v \in V$ and then $\deg(nv) = n \quad \forall n \in \mathbb{Z}$). So it suffices to

show exactness at $\mathbb{Z}V$. A \mathbb{Z} -basis for $\ker \partial$ is $\{v-s : v \in V - \{s\}\}$

where s is any fixed vertex. To see this: suppose $D = \sum a_{uv}$ and

$$\deg D = \sum_{v \in V} a_{sv} = 0 \Rightarrow a_s = -\sum_{v \in V - \{s\}} a_{sv}. \text{ Therefore, } D - \sum_{v \in V - \{s\}} a_{sv}(v-s) = 0.$$

Also $\{v-s : v \in V - \{s\}\}$ is clearly linearly independent.

It is also clear from the definition of ∂ that $\text{im}(\partial) \subseteq \ker(\deg)$.

So it remains to show that each $v-s \in \text{im}(\partial)$.

Let $v \in V \setminus \{s\}$. Since G is connected, there is a path $s = v_0, v_1, v_2, \dots, v_k = v$ to s in G . Let e_i be the edge connecting v_i and v_{i+1} . Let $s_i = 1$ if $e_i = (v_i, v_{i+1})$ and $s_i = -1$ if $e_i = (v_{i+1}, v_i)$. Then $\partial(\sum s_i e_i) = v - s$.

b) Fix a forest F for G and let $v \in V$. Show

$$c_v^a = \sum_{e \in F : e^- = v} c_e^a - \sum_{e \in F : e^+ = v} c_e^*$$

PF/ We may assume G is connected since no edges from a component of G not containing v appear in the above equation. Thus, we may assume F is a tree. Say there are k edges of F incident on v . Removing these from F disconnects F into $k+1$ components, including $\{v\}$. The edges appearing in the c_e^*

exactly the edges f connecting a pair of these components. If f connects a component to $\{v\}$, it appears in exactly one c_e^* (with the right sign), and if not, it appears in exactly 2 with opposite sign.