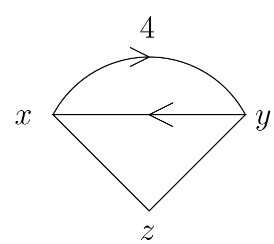
HW 8, due Friday, November 5 *Math 412*

1. Riemann-Roch for directed graphs.

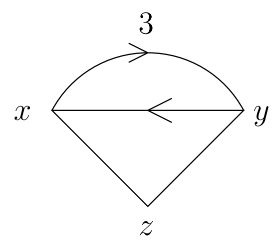
   (a) Suppose Γ is a directed multigraph with trivial critical group. Show there is a unique divisor class K and a unique integer g such that
   $$r(D) - r(K-D) = \deg D + 1 - g$$
   for all divisors D.

   (b) Let Γ be the graph

   The critical group is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. The divisor $A := x - z$ is a generator. Thus, in degree d, the class group has three elements $d\,z$, $A + d\,z$, and $2A + d\,z$. Computation shows that
   $$r(D) = \begin{cases} -1 & \text{if } \deg(D) < 0 \text{ or } (\deg(D) = 0 \text{ and } D \neq 0), \\ 0 & \text{if } D = 0, \\ \deg(D) - 1 & \text{if } \deg(D) > 0. \end{cases}$$
   Show there is a unique divisor class K and a unique integer g such that
   $$r(D) - r(K-D) = \deg(D) + 1 - g,$$
   for all divisors D.

2. Let Γ be the graph



(a) Use the minimal burning script for (Γ, z) to compute a Gröbner basis for the affine toppling ideal of (Γ, z) with respect to degrevlex ordering with $x > y$.

(b) A *minimal Gröbner basis* is computed by first computing any Gröbner basis, then reducing the elements of the basis against each other to ensure that no leading term of one basis element divides any of the leading terms of the other elements. **The** *reduced Gröbner basis* comes from a minimal Gröbner basis by further reducing to ensure that no leading term of a basis elements divides any of the terms of the other elements, then scaling by constant if necessary to make sure that each remaining basis element is monic. Compute the reduced Gröbner basis for the affine toppling ideal of (Γ, z) with respect to the given ordering.

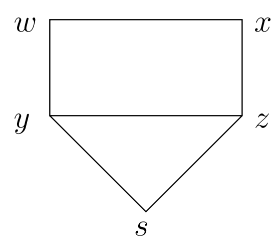
(c) Compute the reduced Gröbner basis for the affine toppling ideal of (Γ, y) with respect to degrevlex ordering with $x > z$.

(d) Compute the reduced Gröbner basis for the affine toppling ideal of (Γ, x) with respect to degrevlex ordering with $y > z$.

3. Let Γ be an undirected graph. Fix a vertex s of Γ. Since the minimal burning script for (Γ, s) is $\vec{1}$, the following gives a Gröbner basis for the affine toppling ideal with respect to a toppling order:

$$\mathcal{G} = \{x^{\widetilde{\Delta}(U)^+} - x^{\widetilde{\Delta}(U)^-} : U \subseteq \widetilde{V}\}.$$

So the elements of the Gröbner basis are indexed by the cuts, (U, U^c). A cut (U, U^c) is *well-connected* if the subgraphs induced by U and U^c are both connected. A theorem—due to Cori, Rossin, and Salvy— states that a minimal Gröbner basis for the affine toppling ideal is obtained by ranging over just the well-connected cuts, (U, U^c) (which we identify with (U^c, U) for this purpose). Illustrate this theorem by computing the Gröbner basis for the following graph, using all cuts, then showing that those elements corresponding to cuts that are not well-connected are superfluous:



Choose $w > x > y > z$ for the sandpile toppling order. (The indeterminate s does not appear in the affine toppling ideal.)

4. Let $\Gamma = (V, E)$ be an undirected connected graph. A *disconnecting set* of edges is a collection $F \subseteq E$ such that $G \setminus F$ (the graph obtained from Γ by removing the edges F) has more than one component. A *cut* is an ordered partition, (U, U^c), of V. The edges having one vertex in U and the other in U^c forms the *cut set* determined by the partition.

   (a) Every non-empty cut set forms a disconnecting set. Show that the converse is not true.

   (b) Show that every minimal (with respect to inclusion) disconnecting set is a cut set.

   (c) A *bond* is a cut set that is minimal with respect to inclusion. Show that a nonempty cut set F is a bond iff $\Gamma \setminus F$ has exactly two components.

   (d) We identified each cut (U, U^c) with an element $c_U^* \in \mathbb{Z}E$ and defined the cut space, $\mathcal{C}^*$, to be the span of these cuts. Show that the bonds span $\mathcal{C}^*$. Do they form a basis?

5. Suppose $G = (V, E)$ is an oriented, unweighted graph.

   (a) If G is connected, prove that the following sequence is exact

$$0 \to \mathcal{C} \to \mathbb{Z}E \xrightarrow{\partial} \mathbb{Z}V \xrightarrow{\deg} \mathbb{Z} \to 0.$$

   (We have already shown exactness at $\mathcal{C}$ and $\mathbb{Z}E$.)

(b) Fix a forest, F. For each $e \in F$, we have defined the cut-set $c_e^* \in \mathbb{Z}E$. For each $v \in V$, we also have the cut-set c_v^* corresponding to the cut, $(v, V \setminus v)$. Prove that

$$c_v^* = \sum_{e \in F : e^- = v} c_e^* - \sum_{e \in F : e^+ = v} c_e^*.$$