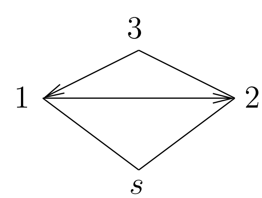
HW 7, due Friday, October 29 *Math 412*

1. Let Γ be a directed multigraph.
   (a) Show that $D \in \operatorname{div}(\Gamma)$ is alive iff $|D_{\max} - D| = \emptyset$.
   (b) Define the nonspecial divisors, $\mathcal{N}$, to be the maximal (with respect to component-wise comparision) divisors with empty linear system. Define a divisor to be *alive* or *minimally alive* just as we have down for Eulerian graphs. Prove that the following are equivalent for $D \in \operatorname{div}(\Gamma)$:
      i. $D \in \mathcal{N}$
      ii. $D_{\max} - D$ is minimally alive.
      iii. $|D| = \emptyset$ and $|D + v| \neq \emptyset$ for all $v \in V_\Gamma$.
2. Consider the graph, Γ:



(The only directed edges are $1 \to 2$ and $3 \to 1$.) For convenience, here is a reminder of important exact sequences:

$$\mathbb{Z}V \xrightarrow{\Delta} \mathbb{Z}V \to \operatorname{Cl}(\Gamma) \to 0$$

$$\mathbb{Z}V \xrightarrow{\Delta} \mathbb{Z}V_0 \to \operatorname{critical}(\Gamma) \to 0$$
$$0 \to \mathbb{Z}/\tilde{\tau}_s\mathbb{Z} \to \mathcal{S}(\Gamma, s) \to \operatorname{critical}(\Gamma) \to 0.$$

We also have the isomorphism

$$\operatorname{Cl}(\Gamma) \xrightarrow{\approx} \mathbb{Z} \oplus \operatorname{critical}(\Gamma)$$
$$[D] \mapsto (\deg(D), D - (\deg D)\, s).$$

Here, $\tilde{\tau}_s$ is the s-coordinate of the generator for the kernel of Δ (which can also be described in terms of directed trees).

(a) Calculate the recurrents of Γ and draw them in a poset lattice with respect to component-wise ordering: $c \geq \tilde{c}$ if $c(v) \geq \tilde{c}(v)$ for all $v \in V_\Gamma$. You should see that not all of the minimal recurrents have the same degree.

(b) Identify the sandpile group as an abelian group.

(c) Identify the class group as an abelian group.

(d) Find the generator of the kernel of the Laplacian and verify that the third exact sequence listed above, relating the sandpile group to the critical group, is satisfied.

(e) Find a minimal alive divisor on this graph and find a minimal length cycle obtained by firing all unstable vertices at each step.

(f) Find an element with full support in the linear system of the minimal alive divisor you found.

(g) What are the nonspecial divisors (as defined in the problem 1)?

(h) In the case of an undirected graph, we saw that a divisor D is nonspecial iff there exists a maximal superstable c such that $D \sim c - s$. Is that true here?

(i) Does RR1 hold: for all $D \in \mathrm{div}(\Gamma)$ there exists $\nu \in \mathcal{N}$ such that exactly one of $|D|$ and $|\nu - D|$ is empty?

(j) Is there any hope for something like the canonical divisor existing?

3. Consider the ideal $I = \langle xy - z^3, y^2 - xz \rangle \subset \mathbb{C}[x, y, z]$.

(a) Compute the reduced Gröbner basis for I with respect to DegLex with $x > y > z$.

(b) Compute the reduced Gröbner basis for I with respect to DegRevLex with $x > y > z$.