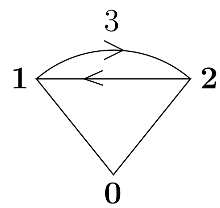
HW 6, due Friday, October 8 *Math 412*

1. Let T be a triangle with vertices $\{0, 1, 2\}$ but with the edge from 1 to 2 having weight 3. Take 0 as the sink.



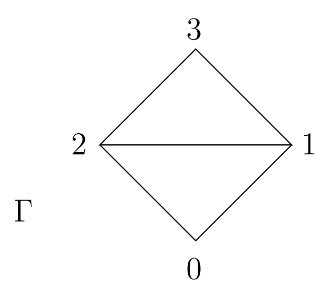
   (a) Find all the recurrents and all the superstables on T.

   (b) For each recurrent c, find $\sigma \geq 0$ such that $c - \widetilde{\Delta}_T \sigma$ is superstable. Can each of these σs be realized as a sequence of legal vertex firings?

2. Let S be a square with vertices $\{0, 1, 2, 3\}$ bit with the edge from 1 to 2 having weight 2. Take 0 as the sink. Repeat the above exercise.

3. State and proof a necessary and sufficient condition that the recurrents and superstables on a sandpile graph $(\Gamma, 0)$ are the same.

4. Consider the following graph



   (a) List all the superstables by degree.

   (b) List all the non-special divisors, $\mathcal{N}$, in standard form (0-reduced).

(c) For each non-special divisor ν, write $K - \nu$ in standard form to see a permutation of $\mathcal{N}$.

(d) Find a representative for each minimal alive divisor modulo $\mathcal{L}$.

5. Parking spaces $\{0, 1, \ldots, n-1\}$ are available consecutively. There are n cars, $C_0, \ldots, C_{n-1}$, and car C_i prefers space p_i. This means that car C_i will pass spaces $p_0, \ldots, p_{i-1}$, then take space p_i if it is available. If space p_i is unavailable, car C_i will take then next available space, if possible. A *parking function* is a function $p\colon \{0, \ldots, n-1\} \to \{0, \ldots, n-1\}$ (which we present as a list $p_0 \cdots p_{n-1}$), that allows every car to park. For instance, the parking functions in the case $n = 2$ are 00, 10, and 01. In the case $n = 3$, the parking functions are 000, 001, 010, 100, 002, 020, 200, 011, 101, 110, 012, 021, 102, 120, 201, and 210.

   (a) Given any function $p\colon \{0, \ldots, n-1\} \to \{0, \ldots, n-1\}$, let

   $$a_0 \leq a_2 \leq \cdots \leq a_{n-1}$$

   be the non-decreasing rearrangement of $p_0, \ldots, p_{n-1}$. Show that p is a parking function iff $a_i \leq i$ for $i = 0, \ldots, n-1$. It then trivially follows that the composition of any permutation of $\{0, \ldots, n-1\}$ with a parking function is again a parking function.

   (b) Show that the parking functions are exactly the superstables for the complete graph K_{n+1} (and hence, the number of labeled trees with n nodes).