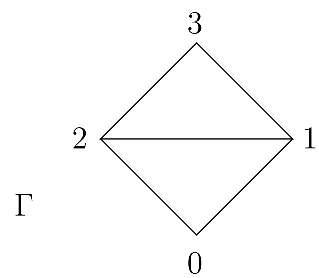
HW 5, due Friday, October 1 *Math 412*

Let Γ be a connected, undirected (finite) graph with genus g and canonical divisor K. Let D be a divisor on Γ.

1. Describe all graphs with genus 0.
2. Describe a family of graphs $\{\Gamma_i\}_{i\geq 0}$ for which the genus of Γ_i is i.
3. Prove that if $\deg D < 0$, then $r(D) = -1$.
4. Prove that if $\deg D = 0$, then $r(D) \in \{-1, 0\}$ with $r(D) = 0$ iff $D \sim 0$.
5. Calculate $\deg(K)$, expressing your answer in terms of the genus, g.
6. Show that if $|D - E| \neq \emptyset$ for all effective divisors E of degree e, then $|D - F| \neq \emptyset$ for all effective divisors F with $\deg F \leq e$.
7. Use the Riemann-Roch theorem to calculate $r(K)$, expressing your answer in terms of the genus, g.
8. Let v be a vertex of Γ.
   (a) Without using the Riemann-Roch theorem, prove that $r(D+v) \leq r(D) + 1$.
   (b) What does Riemann-Roch say about $r(nv)$ for n large?
   (c) Characterize the graphs Γ for which $r(v) = 1$.
9. Let D be an effective divisor. Prove that $r(D) \leq \deg D$ with equality iff $D = 0$ or $g = 0$.
10. We defined the critical group of Γ to be $\mathbb{Z}V_0$ modulo the image of the Laplacian, Δ.
    (a) Show that $\mathrm{Cl}(\Gamma) \approx \mathbb{Z} \oplus \mathrm{critical}(\Gamma)$ where $\mathrm{Cl}(\Gamma)$ is the divisor class group. (Hint: I would write out some exact sequences and use the snake lemma.)
    (b) Given the previous problem, describe all of the elements in the divisor class group having degree d for the graph pictured below: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