HW 4, due Friday, September 24 *Math 412*

1. **Adding a source vertex.** Let (Γ, s) be a sandpile graph.

   (a) Let $k \geq 1$ and choose $v_1, \ldots, v_k \in \tilde{V}_\Gamma$. Form the digraph Γ' obtained from Γ by adding a (source) vertex, u, and directed edges (u, v_i) for $i = 1, \ldots, k$. How do the sizes of the sandpile groups for Γ and Γ' compare?

   i. Compare the sizes of the sandpile groups for (Γ, s) and (Γ', s).

   ii. Show the inclusion $\Gamma \to \Gamma'$ induces the short exact sequence

   $$0 \to \mathbb{Z}\widetilde{V}_\Gamma/\widetilde{\Delta}_\Gamma \xrightarrow{\iota} \mathbb{Z}\widetilde{V}_{\Gamma'}/\widetilde{\Delta}_{\Gamma'} \to \mathbb{Z}/kZ \to 0.$$

   iii. Show that the short exact sequence is not split-exact in general by giving an example for which

   $$\mathbb{Z}\widetilde{V}_{\Gamma'}/\widetilde{\Delta}_{\Gamma'} \not\approx \mathbb{Z}\widetilde{V}_\Gamma/\widetilde{\Delta}_\Gamma \oplus \mathbb{Z}/k\mathbb{Z}.$$

   iv. The short exact sequence says that there is a short exact sequence of sandpile groups

   $$0 \to \mathcal{S}(\Gamma) \to \mathcal{S}(\Gamma') \to \mathbb{Z}/k\mathbb{Z}.$$

   If c is recurrent on Γ, prove that it is also recurrent on Γ'.

   v. If $c = \sum_{v \in V_{\Gamma'}} c_v\, v \in \mathcal{S}(\Gamma')$, show that $\tilde{c} := \sum_{v \in V_\Gamma} c_v\, v \in \mathcal{S}(\Gamma)$ (i.e., show that $\tilde{c}$ is recurrent on Γ). Thus, there is a mapping of *sets* $\tilde{\pi} \colon \mathcal{S}(\Gamma') \to \mathcal{S}(\Gamma)$. Show that it is not, in general, a homomorphism of groups.

   (b) Let b be any configuration on Γ that is equivalent to zero modulo the image of $\widetilde{\Delta}_\Gamma$ (for instance, b could be a burning configuration) and form the digraph Γ' by adding a vertex u and directed edges (u, v) with weight b_v for each $v \in \widetilde{V}_\Gamma$. Show that the mapping $\pi : \mathbb{Z}\widetilde{V}_{\Gamma'} \to \mathbb{Z}\widetilde{V}_\Gamma$ defined, for $v \in \widetilde{V}_{\Gamma'}$, by

   $$\pi(v) = \begin{cases} v & \text{if } v \neq u \\ 0 & \text{if } v = u. \end{cases}$$

induces a splitting of the exact sequence in (1(a)ii). In other words, it gives a mapping

$$\pi\colon \mathbb{Z}\widetilde{V}_{\Gamma'}/\widetilde{\Delta}_{\Gamma'} \to \mathbb{Z}\widetilde{V}_{\Gamma}/\widetilde{\Delta}_{\Gamma}$$

such that $\pi \circ \iota$ is the identity mapping (where ι is as defined in (1(a)ii). It then follows that

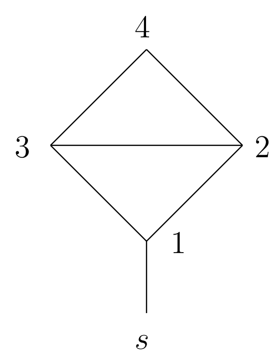
$$\mathbb{Z}\widetilde{V}_{\Gamma'}/\widetilde{\Delta}_{\Gamma'} \approx \mathbb{Z}\widetilde{V}_{\Gamma}/\widetilde{\Delta}_{\Gamma} \oplus \mathbb{Z}/k\mathbb{Z}.$$

Note that this exercise shows that the mapping $\tilde{\pi}$ of part (1b) is a homomorphism of groups give the extra hypothesis that the source is "wired in" to Γ using b.

2. **Trees $\longleftrightarrow$ recurrents.** Let (Γ, s) be an undirected, unweighted, sandpile graph, and let $\mathbf{b} = \widetilde{\Delta}\vec{1}$ be its minimal burning configuration. Order the vertices of Γ arbitrarily (one good choice is to order by distance from the sink, breaking ties arbitrarily). Let $c \in \mathcal{S}(\Gamma, s)$. We use the burning configuration and c to grow a spanning tree on Γ. Let $a = c + b$, and let $\tau = \emptyset \subset E_\Gamma$ be the empty tree. For the first step, add to τ all edges connecting the sink to an unstable vertex of a. Next, let v be the first (with respect to the chosen ordering of the vertices) unstable vertex of a. Replace a by $a - \widetilde{\Delta}v$, and add edges to τ connecting v to vertices that are unstable and are not on any edges already in τ. Repeat: choose the first unstable vertex of a, fire it, update a, add edges to τ from the fired vertex to any unstable vertices that are not already on any edge of τ. Repeat.

   (a) Use properties of the burning configuration to show that when the above process halts, τ is a spanning tree of Γ.

   (b) Apply the process to the following graph

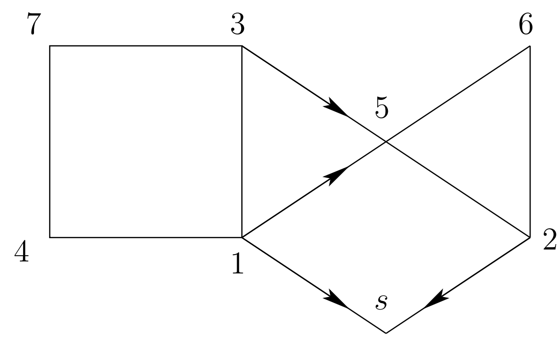


with the indicated vertex ordering and the configurations (i) $c_{\max} = (2, 2, 2, 1)$ and (ii) $c = (2, 2, 1, 1)$.

(c) Prove that the process gives a bijection between recurrents and trees.

(d) Extra credit: Does this method extend to directed graphs? (I do not know the answer.)

3. Find the burning configuration with minimal script for the following graph:



4. Consider the graph Γ in problem 2b. Describe an isomorphism $\mathcal{S}(\Gamma, s) \approx \mathcal{S}(\Gamma, 4)$.