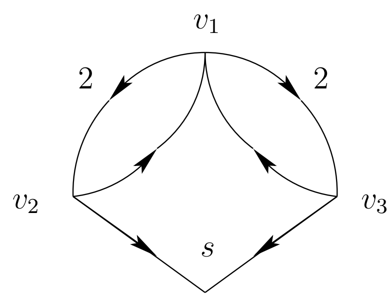
HW 3, due Friday, September 17 *Math 412*

1. Let Γ be the following digraph:



   (a) Let $\widetilde{\Delta}$ be the reduced Laplacian of Γ with respect to its global sink. *By hand*, find 3×3 integer matrices, U, V, invertible over the integers, such that $U\widetilde{\Delta}V = D$ where D is in Smith normal form, thus identifying the sandpile group as

$$\mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z} \times \mathbb{Z}/d_3\mathbb{Z}$$

   with $d_1|d_2|d_3$.

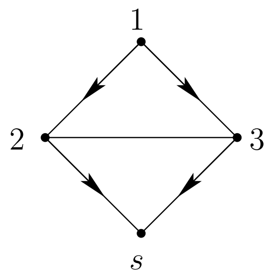
   (b) Find all the spanning trees directed into the sink (checking agreement with the matrix-tree theorem).

2. (a) Show that

$$\det\begin{pmatrix} x & y & y & \dots & y \\ y & x & y & \dots & y \\ \vdots & & \ddots & & \vdots \\ y & y & y & \dots & x \end{pmatrix} = (x-y)^{n-1}(x+(n-1)y).$$

   (b) Use the matrix-tree theorem to prove Cayley's theorem: the number of trees on n labeled vertices is n^{n-2}. (Note that "tree" in this case means a spanning tree of the complete graph on n vertices. The labels are mentioned to distinguish between isomorphic trees, i.e., trees isomorphic as graphs.)

3. Consider the following graph Γ with sink s:

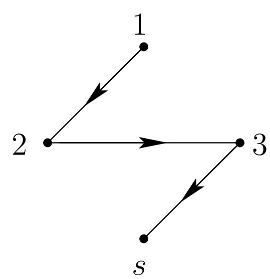


(There is an edge connecting 2 to 3 and an edge connecting 3 to 2.)

(a) Find all recurrent configurations on Γ, indicating the identity configuration.

(b) Use the matrix-tree theorem to determine the number of directed spanning trees directed into the sink.

(c) **Rotor-routers.** Let the sandpile group of Γ operate on the following tree in order to generate all directed spanning trees of Γ:



4. **Reversing edges.** Let (Γ, s) be a sandpile graph with reduced Laplacian $\widetilde{\Delta}$. It is sometimes the case that the transpose, $\widetilde{\Delta}^t$, is the reduced Laplacian of a sandpile graph (Γ', s'). Suppose this is the case.

(a) Let U, V be invertible integer matrices such that $U\widetilde{\Delta}V = D$ where D is the Smith normal from of $\widetilde{\Delta}$. Use U and V to describe an isomorphism of sandpile groups $\mathcal{S}(\Gamma) \to \mathcal{S}(\Gamma')$. (Is there a simpler description of this isomorphism? For instance, in the case of an undirected graph, $\widetilde{\Delta} = \widetilde{\Delta}^t$, and the isomorphism is simply the identity mapping.)

(b) Find an example for which Γ and Γ' are not isomorphic graphs, and explicitly compute this isomorphism.