HW 2, due Friday, September 10 *Math 412*

1. (a) Show that every finite abelian group is the sandpile group of some graph.
   (b) Find two non-isomorphic graphs with the same sandpile group. (An isomorphism of graphs is a bijection of vertex sets inducing a bijection of edge sets.)

2. Prove that the following are equivalent for a sandpile graph $(\Gamma, s)$:
   (a) The configuration $\vec{0}$ is recurrent.
   (b) Every stable configuration is recurrent.
   (c) Every directed cycle of $\Gamma$ must include the sink, $s$.

3. Exhibit a sandpile graph with two stable configurations that are equivalent modulo the image of the reduced Laplacian.

4. Please see notes posted online for last Friday's class. Compare lemma 2 in those notes with lemma 2.14 of "Chip-Firing and Rotor-Routing on Directed Graphs." Is there something wrong with our lemma 2?

5. Let $(\Gamma, s)$ be a sandpile graph, and let $v$ be a nonsink vertex. Let $\widetilde{\Gamma}$ be the graph obtained from $\Gamma$ by adding a loop at $v$.
   (a) Show that $\mathcal{S}(\Gamma, s) \approx \mathcal{S}(\tilde{\Gamma}, s)$ by considering the reduced Laplacian matrices of the two graphs.
   (b) Show that $c \mapsto (c)^o$ gives an isomorphism $\mathcal{S}(\widetilde{\Gamma}, s) \to \mathcal{S}(\Gamma, s)$.

   In our sandpile model, we have taken a configuration $c$ to be *stable* at a vertex $v$ if $c_v < \text{outdeg}(v)$. This exercise shows that we get an equivalent model if we take a vertex to be stable if $c_v \leq \text{outdeg}(v)$.

6. From our homepage and peruse the Sandpile Catalog (look for the link). So far the catalog lists all connected graphs with at most 7 vertices. For each graph, the catalog gives the elementary divisors and the h_vector for the corresponding sandpile group. The h_vector gives the number of recurrents of each size, from largest to smallest, starting

with the maximal stable configuration. List some conjectures (no need for proofs, yet).

7. If you have a computer, download and install Sage from

   `http://sagemath.org/`.

   Download `sandpile.sage` from our class homepage (see the link under the Sage header). Start reading the manual at

   `http://people.reed.edu/~davidp/sand/sage/2.0/html/sandpile.html`.

   Try some of your own examples. After starting Sage, you will need to enter the command

   ```
   sage:  load sandpile.sage
   ```

   (You may need to give the full path name for `sandpile.sage` or put `sandpile.sage` in the same directory as the sage command.)

8. Read the "Review of modules" handout, available at our homepage under "Handouts". Do the exercises in the handout if you have never done them before, but do not turn them in.