HW 1, due Friday, September 3 *Math 412*

I learned about the following, leading to a characterization of the elements of the sandpile group, from exercises for an REU led by Làszlò Babai.

A *semigroup* is a set with an associative binary operation. An *ideal* in a semigroup S is a subset $I \subseteq S$ such that $aI \subseteq I$ and $Ib \subseteq I$ for all $a, b \in S$. A *monoid* is a semigroup with an identity element.

Let S be a semigroup.

1. Prove that the intersection of an arbitrary number (including none and infinitely many) of ideals of S is again an ideal.

2. Prove that the intersection of two nonempty ideals of S is nonempty.

3. Prove that a finite monoid has a unique minimal ideal. Here, *minimal* means nonempty and not properly containing any nonempty ideal.

4. Suppose that S is nonempty and has the property that for all $a, b \in S$, there exist $x, y \in S$ such that $ax = b$ and $ya = b$. Prove that S is a group.

5. Prove that the minimal ideal of a finite *commutative* monoid is a group.

6. Let $\mathcal{M}$ be the monoid of stable configurations on some sandpile graph, Γ. Prove that its minimal ideal is exactly the set of recurrent elements of Γ. (Skip this problem if we don't get to the definition of recurrent elements by Wednesday.) In light of the previous exercise, we get a proof that the set of recurrent elements forms a group.