RIEMANN-ROCH AND ABEL-JACOBI THEORY ON A FINITE GRAPH

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Abstract. It is well-known that a finite graph can be viewed, in many respects, as a discrete analogue of a Riemann surface. In this paper, we pursue this analogy further in the context of linear equivalence of divisors. In particular, we formulate and prove a graph-theoretic analogue of the classical Riemann-Roch theorem. We also prove several results, analogous to classical facts about Riemann surfaces, concerning the Abel-Jacobi map from a graph to its Jacobian. As an application of our results, we characterize the existence or non-existence of a winning strategy for a certain chip-firing game played on the vertices of a graph.

1. Introduction

1.1. **Overview.** In this paper, we explore some new analogies between finite graphs and Riemann surfaces. Our main result is a graph-theoretic analogue of the classical Riemann-Roch theorem. We also study the Abel-Jacobi map S from a graph G to its Jacobian, as well as the higher symmetric powers $S^{(k)}$ of S. We prove, for example, that $S^{(g)}$ is always surjective, and that $S^{(1)}$ is injective when G is 2-edge-connected. These results closely mirror classical facts about the Jacobian of a Riemann surface. As an application of our results, we characterize the existence or non-existence of a winning strategy for a certain chip-firing game played on the vertices of a graph.

The paper is structured as follows. In §1, we provide all of the relevant definitions and state our main results. The proof of the Riemann-Roch theorem for graphs occupies §2-3. In §4, we study the injectivity and surjectivity of $S^{(k)}$ for $k \geq 1$, and explain the connection with the

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chip-firing game. Related results and further questions are discussed in §5. The paper concludes with two appendices. In Appendix A, we provide the reader with a brief summary of some classical results about Riemann surfaces, and in Appendix B, we discuss the graph-theoretic analogue of Abel's theorem proved in [2].

1.2. **Notation and Terminology.** Throughout this paper, a *Riemann surface* will mean a compact, connected one-dimensional complex manifold, and a *graph* will mean a finite, unweighted multigraph having no loop edges. All graphs in this paper are assumed to be connected. We denote by $V(G)$ and $E(G)$, respectively, the set of vertices and edges of G. We will simply write G instead of $V(G)$ when there is no danger of confusion. Also, we write $E_v = E_v(G)$ for the set of edges incident to a given vertex v.

For $k \geq 2$, a graph G is called *k-edge-connected* if $G - W$ is connected for every set W of at most $k-1$ edges of G. (By convention, we consider the trivial graph having one vertex and no edges to be k-edge-connected for all k.) Alternatively, define a *cut* to be the set of all edges connecting a vertex in V_1 to a vertex in V_2 for some partition of $V(G)$ into disjoint non-empty subsets V_1 and V_2. Then G is k-edge-connected if and only if every cut has size at least k.

If $A \subseteq V(G)$, we denote by $\chi_A : V(G) \to \{0, 1\}$ the characteristic function of A.

1.3. **The Jacobian of a finite graph.** Let G be a graph, and choose an ordering $\{v_1, \ldots, v_n\}$ of the vertices of G. The *Laplacian matrix* Q associated to G is the $n \times n$ matrix $Q = D - A$, where D is the diagonal matrix whose $(i,i)^{\text{th}}$ entry is the degree of vertex v_i, and A is the *adjacency matrix* of the graph, whose $(i,j)^{\text{th}}$ entry is the number of edges connecting v_i and v_j. Since loop edges are not allowed, the $(i,i)^{\text{th}}$ entry of A is zero for all i. It is well-known and easy to verify that Q is symmetric, has rank $n-1$, and that the kernel of Q is spanned by the vector whose entries are all equal to 1 (see [3, 10, 16]).

Let $\mathrm{Div}(G)$ be the free abelian group on the set of vertices of G. We think of elements of $\mathrm{Div}(G)$ as formal integer linear combinations of elements of $V(G)$, and write an element $D \in \mathrm{Div}(G)$ as $\sum_{v \in V(G)} a_v(v)$, where each a_v is an integer. By analogy with the Riemann surface case, elements of $\mathrm{Div}(G)$ are called *divisors* on G.

For convenience, we will write $D(v)$ for the coefficient a_v of (v) in D.

There is a natural partial order on the group $\mathrm{Div}(G)$: we say that $D \geq D'$ if and only if $D(v) \geq D'(v)$ for all $v \in V(G)$. A divisor

$E \in \mathrm{Div}(G)$ is called *effective* if $E \geq 0$. We write $\mathrm{Div}_+(G)$ for the set of all effective divisors on G.

The *degree function* $\deg : \mathrm{Div}(G) \to \mathbb{Z}$ is defined by $\deg(D) = \sum_{v \in V(G)} D(v)$.

Remark 1.1. Note that the definitions of the partial order $\geq$, the space $\mathrm{Div}_+(G)$, and the map deg make sense when $V(G)$ is replaced by an arbitrary set X. This observation will be used in §2 when we formulate an abstract "Riemann-Roch Criterion".

We let $\mathcal{M}(G) = \mathrm{Hom}(V(G), \mathbb{Z})$ be the abelian group consisting of all integer-valued functions on the vertices of G. One can think of $\mathcal{M}(G)$ as analogous to the field $\mathcal{M}(X)$ of meromorphic functions on a Riemann surface X (though it is actually more like the abelian group $\{\log|f| \ : \ f \in \mathcal{M}(X)^*\}$, see Remark 1.4).

Using our ordering of the vertices, we obtain isomorphisms between $\mathrm{Div}(G), \mathcal{M}(G)$, and the space of $n \times 1$ column vectors having integer coordinates. We write $[D]$ (resp. $[f]$) for the column vector corresponding to $D \in \mathrm{Div}(G)$ (resp. $f \in \mathcal{M}(G)$). The *Laplacian operator* $\Delta : \mathcal{M}(G) \to \mathrm{Div}(G)$ is given by the formula

$$\Delta(f) = \sum_{v \in V(G)} \Delta_v(f)(v) \ ,$$

where

$$\begin{aligned} \Delta_v(f) &= \deg(v) f(v) - \sum_{e = wv \in E_v} f(w) \\ &= \sum_{e = wv \in E_v} (f(v) - f(w)) \ . \end{aligned}$$

In terms of matrices, it follows from the definitions that

$$[\Delta(f)] = Q[f] \ .$$

Remark 1.2. The fact that Q is a symmetric matrix is equivalent to the fact that Δ is self-adjoint with respect to the bilinear pairing $\langle f, D \rangle = \sum_{v \in V(G)} f(v) D(v)$ on $\mathcal{M}(G) \times \mathrm{Div}(G)$. This is the graph-theoretic analogue of the *Weil reciprocity theorem* on a Riemann surface (see p. 242 of [17] and Remark 1.4 below).

We define the subgroup $\mathrm{Div}^0(G)$ of $\mathrm{Div}(G)$ consisting of *divisors of degree zero* to be the kernel of deg, i.e.,

$$\mathrm{Div}^0(G) = \{D \in \mathrm{Div}(G) \ : \ \deg(D) = 0\} \ .$$

More generally, for each $k \in \mathbb{Z}$ we define $\mathrm{Div}^k(G) = \{D \in \mathrm{Div}(G) \ : \ \deg(D) = k\}$, and $\mathrm{Div}^k_+(G) = \{D \in \mathrm{Div}(G) \ : \ D \geq 0 \text{ and } \deg(D) = k\}$. The set $\mathrm{Div}^1_+(G)$ is canonically isomorphic to $V(G)$.

We also define the subgroup $\mathrm{Prin}(G)$ of $\mathrm{Div}(G)$ consisting of *principal divisors* to be the image of $\mathcal{M}(G)$ under the Laplacian operator, i.e.,

$$\mathrm{Prin}(G) := \Delta(\mathcal{M}(G)) \ . \tag{1.3}$$

It is easy to see that every principal divisor has degree zero, so that $\mathrm{Prin}(G)$ is a subgroup of $\mathrm{Div}^0(G)$.

Remark 1.4. The classical motivation for (1.3) is that the divisor of a nonzero meromorphic function f on a Riemann surface X can be recovered from the extended real-valued function $\log|f|$ using the (distributional) Laplacian operator Δ. More precisely if $\Delta(\varphi)$ is defined so that

$$\int_X \psi \, \Delta(\varphi) = \int_X \varphi \, \Delta(\psi)$$

for all suitably smooth test functions $\psi : X \to \mathbb{R}$, where $\Delta(\psi)$ is given in local coordinates by the formula

$$\Delta(\psi) = \frac{1}{2\pi} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) dx \wedge dy \ ,$$

then

$$\Delta(\log|f|) = \sum_{P \in X} \mathrm{ord}_P(f) \delta_P \ .$$

In other words, the divisor of f can be identified with the Laplacian of $\log|f|$.

Following [2] and [31], we define the group $\mathrm{Jac}(G)$, called the *Jacobian* of G, to be the corresponding quotient group:

$$\mathrm{Jac}(G) = \frac{\mathrm{Div}^0(G)}{\mathrm{Prin}(G)} \ . \tag{1.5}$$

As shown in [2], $\mathrm{Jac}(G)$ is a finite abelian group whose order $\kappa(G)$ is the number of *spanning trees* in G. (This is a direct consequence of Kirchhoff's famous *Matrix-Tree Theorem*, see §14 of [4].) The group $\mathrm{Jac}(G)$ is a discrete analogue of the Jacobian of a Riemann surface. We will write $[D]$ for the class in $\mathrm{Jac}(G)$ of a divisor $D \in \mathrm{Div}^0(G)$. (There should not be any confusion between this notation and our similar notation for the column vector associated to a divisor.)

In [2], the group $\mathrm{Jac}(G)$ is called the *Picard group*, and denoted $\mathrm{Pic}(G)$, and the term *Jacobian* is reserved for an *a priori* different group denoted $J(G)$. However, as shown in Proposition 7 of [2], the two

groups are canonically isomorphic. The isomorphism $\mathrm{Pic}(G) \cong J(G)$ is the graph-theoretic analogue of *Abel's theorem* (see Theorem VIII.2.2 of [28]).

1.4. **The Abel-Jacobi map from a graph to its Jacobian.** If we fix a base point $v_0 \in V(G)$, we can define the *Abel-Jacobi map* $S_{v_0} : G \to \mathrm{Jac}(G)$ by the formula

$$S_{v_0}(v) = [(v) - (v_0)] \ . \tag{1.6}$$

We also define, for each natural number $k \geq 1$, a map $S_{v_0}^{(k)} : \mathrm{Div}_+^k(G) \to \mathrm{Jac}(G)$ by

$$S_{v_0}^{(k)}((v_1) + \cdots + (v_k)) = S_{v_0}(v_1) + S_{v_0}(v_2) + \cdots + S_{v_0}(v_k) \ .$$

The map S_{v_0} can be characterized by the following universal property (see §3 of [2]). A map $\varphi : G \to A$ from $V(G)$ to an abelian group A is called *harmonic* if for each $v \in G$, we have

$$\deg(v) \cdot \varphi(v) = \sum_{e=wv \in E_v} \varphi(w) \ .$$

Then S_{v_0} is universal among all harmonic maps from G to abelian groups sending v_0 to 0, in the sense that if $\varphi : G \to A$ is any such map, then there is a unique group homomorphism $\psi : \mathrm{Jac}(G) \to A$ such that $\varphi = \psi \circ S_{v_0}$.

Let $g = |E(G)| - |V(G)| + 1$ be the *genus*[1] of G, which is the number of linearly independent cycles of G, or equivalently, the first Betti number of G (i.e., the dimension of $H_1(G, \mathbb{R})$).

We write S instead of S_{v_0} when the base point v_0 is understood. In §4, we will prove:

Theorem 1.7. *The map $S^{(k)}$ is surjective if and only if $k \geq g$.*

The surjectivity of $S^{(g)}$ is the graph-theoretic analogue of a classical result about Riemann surfaces known as Jacobi's Inversion Theorem (see p. 235 of [17]). For a Riemann surface X, it is clear that $S^{(g-1)} : X^{(g-1)} \to \mathrm{Jac}(X)$ is not surjective, since $\dim S^{(g-1)} = g - 1 < \dim \mathrm{Jac}(X) = g$.

As a complement to Theorem 1.7, we will also precisely characterize the values of k for which $S^{(k)}$ is injective:

[1] In graph theory, the term "genus" is traditionally used for a different concept, namely, the smallest genus (i.e., first Betti number) of any surface in which the graph can be embedded, and the integer g is called the "cyclomatic number" of G. We call g the genus of G in order to highlight the analogy with Riemann surfaces.

Theorem 1.8. *The map $S^{(k)}$ is injective if and only if G is $(k+1)$-edge-connected*

For 2-edge-connected graphs, Theorem 1.8 is the analogue of the well-known fact that the Abel-Jacobi map from a Riemann surface X to its Jacobian is injective if and only if X has genus at least 1. (See Proposition VIII.5.1 of [28].)

1.5. **Chip-firing games on graphs.** There have been a number of papers devoted to "chip-firing games" played on the vertices of a graph; see, e.g., [5, 8, 9, 16, 25, 26, 39, 42]. In this paper, as an application of Theorem 1.7, we study a new chip firing game with some rather striking features.

Our chip-firing game, like the one considered by Biggs in [5] (see also §31-32 of [4]), is most conveniently stated using "dollars" rather than chips. Let G be a graph, and consider the following game of "solitaire" played on the vertices of G. The initial configuration of the game assigns to each vertex v of G an integer number of dollars. Such a configuration can be identified with a divisor $D \in \mathrm{Div}(G)$. A vertex which has a negative number of dollars assigned to it is said to be in *debt*. A *move* consists of a vertex v either borrowing one dollar from each of its neighbors or giving one dollar to each of its neighbors. Note that any move leaves the total number of dollars unchanged. The object of the game is to reach, through a sequence of moves, a configuration in which no vertex is in debt. We will call such a configuration a *winning position*, and a sequence of moves which achieves such a configuration a *winning strategy*.

As before, we let $g = |E(G)| - |V(G)| + 1$. In §4.2, we will prove the following result by showing that it is equivalent to Theorem 1.7:

Theorem 1.9. *Let $N = \deg(D)$ be the total number of dollars present at any stage of the game.*

1. *If $N \geq g$, then there is always a winning strategy.*
2. *If $N \leq g-1$, then there is always an initial configuration for which no winning strategy exists.*

See §5.5 for a discussion of the relationship between our chip-firing game and the one studied by Björner, Lovász, and Shor in [9], and see §5.6 for a discussion of the relationship between our chip-firing game and the dollar game of Biggs.

1.6. **Linear systems and the Riemann-Roch theorem.** We define an equivalence relation $\sim$ on the group $\mathrm{Div}(G)$ by declaring that $D \sim D'$ if and only if $D - D' \in \mathrm{Prin}(G)$. Borrowing again from the theory

of Riemann surfaces, we call this relation *linear equivalence*. Since a principal divisor has degree zero, it follows that linearly equivalent divisors have the same degree. Note that by (1.5), the Jacobian of G is the set of linear equivalence classes of degree zero divisors on G.

For $D \in \mathrm{Div}(G)$, we define the *linear system associated to* D to be the set $|D|$ of all effective divisors linearly equivalent to D:

$$|D| = \{E \in \mathrm{Div}(G) \ : \ E \geq 0, \ E \sim D\} \ .$$

As we will see in §4.2, it follows from the definitions that two divisors D and D' on G are linearly equivalent if and only if there is a sequence of moves taking D to D' in the chip firing game described in §1.5. It follows that there is a winning strategy in the chip-firing game whose initial configuration corresponds to D if and only if $|D| \neq \emptyset$.

We define the *dimension* $r(D)$ of the linear system $|D|$ by setting $r(D)$ equal to -1 if $|D| = \emptyset$, and then declaring that for each integer $s \geq 0$, $r(D) \geq s$ if and only if $|D - E| \neq \emptyset$ for all effective divisors E of degree s. It is clear that $r(D)$ depends only on the linear equivalence class of D.

Remark 1.10. By Lemma 4.3 below, we have $r(D) \geq 0$ if and only if there is a winning strategy in the chip firing game with initial configuration D, $r(D) \geq 1$ if and only if there is still a winning strategy after subtracting one dollar from any vertex, etc.

The *canonical divisor* on G is the divisor K given by

$$K = \sum_{v \in V(G)} (\deg(v) - 2)\,(v) \ . \tag{1.11}$$

Since the sum over all vertices v of $\deg(v)$ equals twice the number of edges in G, we have $\deg(K) = 2|E(G)| - 2|V(G)| = 2g - 2$.

We can now state a graph-theoretic analogue of the Riemann-Roch theorem (see Theorem VI.3.11 of [28]). The proof will be given in §3.

Theorem 1.12 (Riemann-Roch for Graphs)**.** *Let G be a graph, and let D be a divisor on G. Then*

$$r(D) - r(K - D) = \deg(D) + 1 - g \ .$$

Remark 1.13. (i) Our definition of $r(D)$ agrees with the usual definition of $r(D)$ as $\dim L(D) - 1$ in the Riemann surface case (see, e.g., p. 250 of [17] or §III.8.15 of [13]).

(ii) One must be careful, however, not to rely too much on intuition from the Riemann surface case when thinking about the quantity $r(D)$ for divisors on graphs. For example, for Riemann surfaces one has

$r(D) = 0$ if and only if $|D|$ contains exactly one element, but neither implication is true in general for graphs. For example, consider the canonical divisor K on a graph G with two vertices v_1 and v_2 connected by m edges. Then clearly $r(K) \geq m-2$, and in fact we have $r(K) = m-2$. (This can be proved directly, or deduced as a consequence of Theorem 1.12.) However, $|K| = \{K\}$ as

$$D \sim K \;\Leftrightarrow\; \exists\, i \in \mathbb{Z} \;:\; D = (m-2+im)(v_1) + (m-2-im)(v_2)\ .$$

To see that the other implication also fails, consider a graph G with $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_3v_1\}$, and $D = 2(v_4) \in \mathrm{Div}(G)$. Then $(v_3) + (v_5) \in |D|$, but it follows from Lemma 3.2 (or can be verified directly) that $|D - (v_1)| = \emptyset$, and therefore $r(D) = 0$.

(iii) The set $L(D) := \{f \in \mathcal{M}(G) \;:\; \Delta(f) \geq -D\}$ is not a vector space, so one cannot just define the number $r(D)$ as $\dim L(D) - 1$ as in the classical case. This should not be surprising, since elements of $L(D)$ are analogous to functions of the form $\log|f|$ with f a nonzero meromorphic function on a Riemann surface X. On the other hand, $L(D) \cup \{\infty\}$ is naturally a finitely generated semimodule over the tropical semiring $(\mathbb{N} \cup \{\infty\}, \min, +)$ (see §2.4 of [15]), and there is a natural notion in this context for the *dimension* of $L(D)$ (see Corollary 95 in [15]). However, examples like the ones above show that the tropical dimension of $L(D)$ is *not* the same as $r(D) + 1$, and does not obey Theorem 1.12.

2. A Riemann-Roch criterion

In this section, we formulate an abstract criterion giving necessary and sufficient conditions for the Riemann-Roch formula $r(D) - r(K - D) = \deg(D) + 1 - g$ to hold, where $r(D)$ is defined in terms of an equivalence relation on an arbitrary free abelian group. This result, which is purely combinatorial in nature, will be used in §3 in our proof of the Riemann-Roch theorem for graphs.

The general setup for our result is as follows.

Let X be a non-empty set, and let $\mathrm{Div}(X)$ be the free abelian group on X. As usual, elements of $\mathrm{Div}(X)$ are called *divisors* on X, divisors E with $E \geq 0$ are called *effective*, and for each integer d, we denote by $\mathrm{Div}_+^d(X)$ the set of effective divisors of degree d on X.

Let $\sim$ be an equivalence relation on $\mathrm{Div}(X)$ satisfying the following two properties:

(E1) If $D \sim D'$ then $\deg(D) = \deg(D')$.
(E2) If $D_1 \sim D_1'$ and $D_2 \sim D_2'$, then $D_1 + D_1' \sim D_2 + D_2'$.

For each $D \in \mathrm{Div}(X)$, define $|D| = \{E \in \mathrm{Div}(G) \ : \ E \geq 0, \ E \sim D\}$, and define the function $r : \mathrm{Div}(X) \to \{-1, 0, 1, 2, \ldots\}$ by declaring that for each integer $s \geq 0$,

$$r(D) \geq s \iff |D - E| \neq \emptyset \, \forall \, E \in \mathrm{Div}(X) \ : \ E \geq 0 \text{ and } \deg(E) = s \ .$$

Note that the above equivalence is true for all integers s. It is easy to see that $r(D) = -1$ if $\deg(D) < 0$, and if $\deg(D) = 0$ then $r(D) = 0$ if $D \sim 0$ and $r(D) = -1$ otherwise.

Lemma 2.1. *For all $D, D' \in \mathrm{Div}(X)$ such that $r(D), r(D') \geq 0$, we have $r(D + D') \geq r(D) + r(D')$.*

Proof. Let $E_0 = (x_1) + \cdots + (x_{r(D)+r(D')})$ be an arbitrary effective divisor of degree $r(D) + r(D')$, and let $E = (x_1) + \cdots + (x_{r(D)})$ and $E' = (x_{r(D)+1}) + \cdots + (x_{r(D)+r(D')})$. Then $|D - E|$ and $|D' - E'|$ are non-empty, so that $D - E \sim F$ and $D' - E' \sim F'$ with $F, F' \geq 0$. It follows that $(D + D') - (E + E') = (D + D') - E_0 \sim F + F' \geq 0$, and thus $r(D + D') \geq r(D) + r(D')$. □

Let g be a nonnegative integer, and define

$$\mathcal{N} = \{D \in \mathrm{Div}(X) \ : \ \deg(D) = g - 1 \text{ and } |D| = \emptyset\} \ .$$

Finally, let K be an element of $\mathrm{Div}(X)$ having degree $2g-2$. The following theorem gives necessary and sufficient conditions for the Riemann-Roch formula to hold for elements of $\mathrm{Div}(X)/\sim$.

Theorem 2.2. *Define $\epsilon : \mathrm{Div}(X) \to \mathbb{Z}/2\mathbb{Z}$ by declaring that $\epsilon(D) = 0$ if $|D| \neq \emptyset$ and $\epsilon(D) = 1$ if $|D| = \emptyset$. Then the Riemann-Roch formula*

$$r(D) - r(K - D) = \deg(D) + 1 - g \tag{2.3}$$

holds for all $D \in \mathrm{Div}(X)$ if and only if the following two properties are satisfied:

(RR1) *For every $D \in \mathrm{Div}(X)$, there exists $\nu \in \mathcal{N}$ such that $\epsilon(D) + \epsilon(\nu - D) = 1$.*

(RR2) *For every $D \in \mathrm{Div}(X)$ with $\deg(D) = g - 1$, we have $\epsilon(D) + \epsilon(K - D) = 0$.*

Remark 2.4. (i) Property (RR2) is equivalent to the assertion that $r(K) \geq g - 1$. Indeed, if (RR2) holds then for every effective divisor E of degree $g - 1$, we have $|K - E| \neq \emptyset$, which means that $r(K) \geq g - 1$. Conversely, if $r(K) \geq g-1$ then $\epsilon(K-E) = \epsilon(E) = 0$ for every effective divisor E of degree $g - 1$. Therefore $\epsilon(D) = 0$ implies $\epsilon(K - D) = 0$. By symmetry, we obtain $\epsilon(D) = 0$ if and only if $\epsilon(K - D) = 0$, which is equivalent to (RR2).

(ii) When the Riemann-Roch formula (2.3) holds, we automatically have $r(K) = g - 1$.

Remark 2.5. (i) When X is a Riemann surface and $\sim$ denotes linear equivalence of divisors, then one can show independently of the Riemann-Roch theorem that $r(K) = g - 1$, i.e., that the vector space of holomorphic 1-forms on X is g-dimensional. Thus one can prove directly that (RR2) holds. We do not know if there is a direct proof of (RR1) which does not make use of Riemann-Roch, but if so, one could deduce the classical Riemann-Roch theorem from it using Theorem 2.2.

(ii) Divisors of degree $g - 1$ on a Riemann surface X which belong to $\mathcal{N}$ are classically referred to as *non-special* (which explains our use of the symbol $\mathcal{N}$).

Before giving the proof of Theorem 2.2, we need a couple of preliminary results. The first is the following simple lemma, whose proof is left to the reader.

Lemma 2.6. *Suppose $\psi : A \to A'$ is a bijection between sets, and that $f : A \to \mathbb{Z}$ and $f' : A' \to \mathbb{Z}$ are functions which are bounded below. If there exists a constant $c \in \mathbb{Z}$ such that*

$$f(a) - f'(\psi(a)) = c$$

for all $a \in A$, then

$$\min_{a \in A} f(a) - \min_{a' \in A'} f'(a') = c \ .$$

If $D = \sum_i a_i(x_i) \in \mathrm{Div}(X)$, we define

$$\deg^+(D) = \sum_{a_i \geq 0} a_i \ .$$

The key observation needed to deduce (2.3) from (RR1) and (RR2) is the following alternate characterization of the quantity $r(D)$:

Lemma 2.7. *If (RR1) holds then for every $D \in \mathrm{Div}(X)$ we have*

$$r(D) = \left(\min_{\substack{D' \sim D \\ \nu \in \mathcal{N}}} \deg^+(D' - \nu) \right) - 1 \ . \tag{2.8}$$

Proof. Let $r'(D)$ denote the right-hand side of (2.8). If $r(D) < r'(D)$, then there exists an effective divisor E of degree $r'(D)$ for which $r(D - E) = -1$. By (RR1), this means that there exists a divisor $\nu \in \mathcal{N}$ and an effective divisor E' such that $\nu - D + E \sim E'$. But then $D' - \nu = E - E'$ for some divisor $D' \sim D$, and thus

$$\deg^+(D' - \nu) - 1 \leq \deg(E) - 1 = r'(D) - 1 \ ,$$

contradicting the definition of $r'(D)$. It follows that $r(D) \geq r'(D)$.

Conversely, if we choose divisors $D' \sim D$ and $\nu \in \mathcal{N}$ achieving the minimum in (2.8), then $\deg^+(D' - \nu) = r'(D) + 1$, and therefore there are effective divisors E, E' with $\deg(E) = r'(D) + 1$ such that $D' - \nu = E - E'$. But then $D - E \sim \nu - E'$, and since $\nu - E'$ is not equivalent to any effective divisor, it follows that $|D - E| = \emptyset$. Therefore $r(D) \leq r'(D)$. □

We can now give the proof of Theorem 2.2.

Proof of Theorem 2.2. We first prove that (2.3) implies (RR1) and (RR2).

Let D be a divisor on X, and let $d = \deg(D)$. Property (RR2) is more or less immediate, since (2.3) implies that if $\deg(D) = g - 1$ then $r(D) = r(K - D)$.

We cannot have $\epsilon(D) = \epsilon(\nu - D) = 0$, or else by Lemma 2.1 we would have $r(\nu) \geq 0$, contradicting the definition of $\mathcal{N}$. As we will see in the next paragraph, $\mathcal{N}$ is non-empty; therefore, to prove (RR1) it suffices to show that if $r(D) = -1$ then $r(\nu - D) \geq 0$ for some $\nu \in \mathcal{N}$.

If $r(D + E) \geq 0$ for all $E \in \mathrm{Div}_+^{g-1-d}(X)$, then (2.3) implies that $r(K - D - E) \geq 0$ for all such E, and therefore $r(K - D) \geq g - 1 - d$. Another application of (2.3) then yields $r(D) = r(K-D)+d+1-g \geq 0$.

Therefore when $r(D) = -1$, there exists an effective divisor E of degree $g - 1 - d$ such that $r(D + E) = -1$. Since $\deg(D + E) = g - 1$, this means that $D + E \in \mathcal{N}$, and therefore $D + E = \nu$ for some $\nu \in \mathcal{N}$. For this choice of ν, we have $r(\nu - D) \geq 0$, which proves (RR1).

We now show that (RR1) and (RR2) imply (2.3). Let $D \in \mathrm{Div}(X)$. For every $\nu \in \mathcal{N}$, property (RR2) implies that $\overline{\nu} := K - \nu$ is also in $\mathcal{N}$. Writing

$$\nu - D' = K - D' - \overline{\nu} \ ,$$

it follows that

$$\begin{aligned} \deg^+(D' - \nu) - \deg^+((K - D') - \overline{\nu}) &= \deg^+(D' - \nu) - \deg^+(\nu - D') \\ &= \deg(D' - \nu) \\ &= \deg(D) + 1 - g \ . \end{aligned}$$

Since the difference $\deg^+(D' - \nu) - \deg^+((K - D') - \overline{\nu})$ has the constant value $\deg(D)+1-g$ for all D' and ν, and since $\overline{\nu} = K-\nu$ runs through all possible elements of $\mathcal{N}$ as ν does, it follows from Lemmas 2.6 and 2.7 that $r(D) - r(K - D) = \deg(D) + 1 - g$ as desired. □

3. Riemann-Roch for graphs

3.1. **G-parking functions and reduced divisors.** In this section, we use the notion of a *G-parking function*, introduced in [33], to define a unique *reduced divisor* in each equivalence class in $\mathrm{Div}(G)$. Reduced divisors will play a key role in our proof of the Riemann-Roch theorem for graphs in the next section. Our reduced divisors are closely related to the "critical configurations" considered by Biggs in [5, 6], as we will explain in Section 5.6.

We now present the relevant definitions. For $A \subseteq V(G)$ and $v \in A$, let $\mathrm{outdeg}_A(v)$ denote the number of edges of G having v as one endpoint and whose other endpoint lies in $V(G) - A$. Select a vertex $v_0 \in V(G)$. We say that a function $f : V(G) - \{v_0\} \longrightarrow \mathbb{Z}$ is a *G-parking function* (relative to the base vertex v_0) if the following two conditions are satisfied:

(P1) $f(v) \geq 0$ for all $v \in V(G) - \{v_0\}$.
(P2) For every non-empty set $A \subseteq V(G) - \{v_0\}$, there exists a vertex $v \in A$ such that $f(v) < \mathrm{outdeg}_A(v)$.

We say that a divisor $D \in \mathrm{Div}(G)$ is *v_0-reduced* if the map $v \mapsto D(v)$, defined for $v \in V(G) - \{v_0\}$, is a G-parking function. In terms of the chip-firing game, a divisor D is v_0-reduced if and only if (1) no vertex $v \neq v_0$ is in debt; and (2) for every non-empty subset A of $V(G) - \{v_0\}$, if all vertices in A were to perform a lending move, some vertex in A would go into debt.

Proposition 3.1. *Fix a base vertex $v_0 \in V(G)$. Then for every $D \in \mathrm{Div}(G)$, there exists a unique v_0-reduced divisor $D' \in \mathrm{Div}(G)$ such that $D' \sim D$.*

Proof. We begin by presenting an informal sketch of the proof that such a divisor D' exists in terms of the chip-firing game. We need to show that any initial configuration can be transformed into a configuration corresponding to a v_0-reduced divisor via a sequence of legal moves. To accomplish this, we first obtain a configuration where no vertex except v_0 is in debt. This can be done, for example, by arranging the vertices in some order, starting with v_0, in such a way that every vertex except for v_0 has a neighbor that precedes it in this order. We then take the vertices out of debt consecutively, starting with the last vertex, by at each step having some neighbor w which precedes the current vertex v in the designated order lend out enough money to take v out of debt.

Once we have obtained a configuration where no vertex other than v_0 is in debt, we enumerate the non-empty subsets $A_1, \ldots, A_s$ of $V(G) - \{v_0\}$. If every vertex of A_1 can give a dollar to each of its neighbors

outside A_1 and remain out of debt, then each vertex of A_1 does so (this is a combination of legal moves in the chip-firing game); otherwise, we move on to the next set A_2, and so on. Once the vertices in some set A_i lend out money, we cycle through the entire procedure again, beginning with A_1. If for each $1 \leq i \leq s$, there is some vertex in A_i which cannot lend a dollar to each of its neighbors outside A_i without going into debt, then the procedure terminates.

Note that v_0 never lends money during this procedure, and so it must stop receiving money at some point. None of the neighbors of v_0 lend money out from this point on, and so they, too, must eventually stop receiving money. Iterating this argument, we see that the entire procedure has to stop. The configuration D' obtained at the end of this process corresponds to a v_0-reduced divisor.

We now formalize the argument presented above. For a vertex $v \in V(G)$, let $d(v)$ denote the length of the shortest path in G between v and v_0. Let $d = \max_{v \in V(G)} d(v)$ and let $S_k = \{v \in V(G) \ : \ d(v) = k\}$ for $0 \leq k \leq d$.

Define the vectors $\mu_1(D) \in \mathbb{Z}^d$ and $\mu_2(D) \in \mathbb{Z}^{d+1}$ by

$$\mu_1(D) = \left(\sum_{\substack{v \in S_d \\ D(v)<0}} D(v), \sum_{\substack{v \in S_{d-1} \\ D(v)<0}} D(v), \ldots, \sum_{\substack{v \in S_1 \\ D(v)<0}} D(v) \right) ,$$

$$\mu_2(D) = \left(\sum_{v \in S_0} D(v), \sum_{v \in S_1} D(v), \ldots, \sum_{v \in S_d} D(v) \right) .$$

Replacing D by an equivalent divisor if necessary, we may assume without loss of generality that

$$\mu_1(D) = \max_{D' \sim D} \mu_1(D') \ \text{ and } \ \mu_2(D) = \max_{\substack{D' \sim D \\ \mu_1(D) = \mu_1(D')}} \mu_2(D'),$$

where the maxima are taken in the lexicographic order. It is easy to see that both maxima are attained. We claim that the resulting divisor D is v_0-reduced.

Suppose $D(v) < 0$ for some vertex $v \neq v_0$. Let v' be a neighbor of v such that $d(v') < d(v)$ and let $D' = D - \Delta(\chi_{\{v'\}})$. Then $D'(v) > D(v)$, and $D'(w) \geq D(w)$ for every w such that $d(w) \geq d(v)$. It follows that $\mu_1(D') > \mu_1(D)$, contradicting the choice of D. Therefore $D(v) \geq 0$ for every $v \in V(G), v \neq v_0$.

Suppose now that for some non-empty subset $A \subseteq V(G) - \{v_0\}$, we have $D(v) \geq \text{outdeg}_A(v)$ for every $v \in A$. Let $D' = D - \Delta(\chi_A)$ and $d_A = \min_{v \in A} d(v)$. We have $D'(v) \geq D(v)$ for all $v \in V(G) - A$

and $D'(v) = D(v) - \mathrm{outdeg}_A(v) \geq 0$ for every $v \in A$. Therefore $\mu_1(D') = \mu_1(D)$, as they are both the zero vector. There must be a vertex $v' \in V(G)$ such that $d(v') < d_A$, and for which v' has a neighbor in A. It follows that $D'(v') > D(v')$, and consequently $\mu_2(D') > \mu_2(D)$, once again contradicting the choice of D. This finishes the proof of the claim.

It remains to show that distinct v_0-reduced divisors cannot be equivalent. Suppose for the sake of contradiction that we are given v_0-reduced divisors D and D' such that $D \sim D'$ and $D \neq D'$. Let $f \in \mathcal{M}(G)$ be a function for which $D' - D = \Delta(f)$. Then f is non-constant, and by symmetry we may assume that $f(v) > f(v_0)$ for some $v \in V(G)$. Let A be the set of all the vertices $v \in V(G)$ for which $f(v)$ is maximal. Then $v_0 \notin A$, and for every $v \in A$ we have

$$0 \leq D(v) = D'(v) - \sum_{e=vw \in E_v} (f(v) - f(w)) \leq D'(v) - \mathrm{outdeg}_A(v) \ .$$

Thus $D'(v) \geq \mathrm{outdeg}_A(v)$ for every $v \in A$, contradicting the assumption that D' is v_0-reduced. □

3.2. **Proof of the Riemann-Roch theorem.** By Theorem 2.2, in order to prove the Riemann-Roch theorem for graphs (Theorem 1.12), it suffices to verify properties (RR1) and (RR2) when $X = G$ is a graph and $\sim$ denotes linear equivalence of divisors. This will be accomplished by analyzing a certain family of divisors of degree $g - 1$ on G.

For each linear (i.e., total) order $<_P$ on $V(G)$, we define

$$\nu_P = \sum_{v \in V(G)} (|\{e = vw \in E(G) \ : \ w <_P v\}| - 1)(v).$$

It is clear that $\deg(\nu_P) = |E(G)| - |V(G)| = g - 1$.

Lemma 3.2. *For every linear order $<_P$ on $V(G)$ we have $\nu_P \in \mathcal{N}$.*

Proof. Let $D \in \mathrm{Div}(G)$ be any divisor of the form $D = \nu_P - \Delta(f)$ for some $f \in \mathcal{M}(G)$. Let $V_f^{\max}$ be the set of vertices $v \in G$ at which f achieves its maximum value, and let u be the minimal element of $V_f^{\max}$ with respect to the order $<_P$. Then $f(w) \leq f(u)$ for all $w \in V(G)$, and if $w <_P u$ then $f(w) < f(u)$. Thus

$$\begin{aligned} D(u) &= (|\{e = uw \in E(G) \ : \ w <_P u\}| - 1) - \sum_{e=uw \in E(G)} (f(u) - f(w)) \\ &= -1 + \sum_{\substack{e=uw \in E(G) \\ u <_P w}} (f(w) - f(u)) + \sum_{\substack{e=uw \in E(G) \\ w <_P u}} (f(w) - f(u) + 1) \\ &\leq -1 \ , \end{aligned}$$

since each term in these sums is non-positive by the choice of u. It follows that ν_P is not equivalent to any effective divisor. □

Theorem 3.3. *For every $D \in \mathrm{Div}(G)$, exactly one of the following holds*

(N1) $r(D) \geq 0$*; or*
(N2) $r(\nu_P - D) \geq 0$ *for some order $<_P$ on $V(G)$.*

Proof. Choose $v_0 \in V(G)$. By Proposition 3.1, we may assume that D is v_0-reduced. We define $v_1, v_2, \ldots, v_{|V(G)|-1}$ inductively as follows. If $v_0, v_1, \ldots, v_{k-1}$ are defined, let $A_k = V(G) - \{v_0, v_1, \ldots, v_{k-1}\}$, and let $v_k \in A_k$ be chosen so that $D(v_k) < \mathrm{outdeg}_{A_k}(v_k)$. Let $<_P$ be the linear order on $V(G)$ such that $v_i <_P v_j$ if and only if $i < j$.

For every $1 \leq k \leq |V(G)| - 1$ we have

$$\begin{aligned} D(v_k) &\leq \mathrm{outdeg}_{A_k}(v_k) - 1 \\ &= |\{e = v_k v_j \in E(G) \ : \ j < k\}| - 1 \\ &= \nu_P(v_k) \ . \end{aligned}$$

If $D(v_0) \geq 0$ then we have $D \geq 0$ and (N1) holds. If, on the other hand, $D(v_0) \leq -1$ then $D \leq \nu_P$ and (N2) holds. Finally, note that if $r(D) \geq 0$ and $r(\nu_P - D) \geq 0$, then $r(\nu_P) \geq 0$ by Lemma 2.1, contradicting Lemma 3.2. □

As a consequence of Lemma 3.2 and Theorem 3.3, we obtain:

Corollary 3.4. *For $D \in \mathrm{Div}(G)$ with $\deg(D) = g - 1$ we have $D \in \mathcal{N}$ if and only if there exists a linear order $<_P$ on $V(G)$ such that $D \sim \nu_P$.*

Proof. It suffices to note that if $\nu_P - D \sim E$ with $E \geq 0$, then $\deg(E) = 0$ and thus $E = 0$, so that $D \sim \nu_P$. □

We can now prove our graph-theoretic version of the Riemann-Roch theorem.

Proof of Theorem 1.12. By Theorem 2.2, it suffices to prove that conditions (RR1) and (RR2) are satisfied.

Let $D \in \mathrm{Div}(G)$, and suppose first that $r(D) \geq 0$. Then for every $\nu \in \mathcal{N}$ we have $r(\nu - D) = -1$, and hence $\epsilon(D) + \epsilon(\nu - D) = 0 + 1 = 1$ and (RR1) holds. Suppose, on the other hand, that $r(D) < 0$. Then by Theorem 3.3, we must have $r(\nu_P - D) \geq 0$ for some order $<_P$ on $V(G)$, and then $\epsilon(D) + \epsilon(\nu_P - D) = 1 + 0 = 1$. As $\nu_P \in \mathcal{N}$ by Lemma 3.2, it follows once again that (RR1) holds.

To prove (RR2), it suffices to show that for every $D \in \mathcal{N}$ we have $K - D \in \mathcal{N}$. By Corollary 3.4, we have $D \sim \nu_P$ for some linear order

$<_P$ on $V(G)$. Let $\bar{P}$ be the *reverse* of P (i.e., $v <_P w \Leftrightarrow w <_{\bar{P}} v$). Then for every $v \in V(G)$, we have

$$\begin{aligned}\nu_P(v) + \nu_{\bar{P}}(v) &= (|\{e = vw \in E(G) \ : \ w <_P v\}| - 1) \\ &\quad + (|\{e = vw \in E(G) \ : \ w <_{\bar{P}} v\}| - 1) \\ &= \deg(v) - 2 = K(v) \ .\end{aligned}$$

Therefore $K - D \sim K - \nu_P = \nu_{\bar{P}} \in \mathcal{N}$. □

3.3. **Consequences of the Riemann-Roch theorem.** As in the Riemann surface case, one can derive a number of interesting consequences from the Riemann-Roch formula. As just one example, we prove a graph-theoretic analogue of Clifford's theorem (see Theorem VII.1.13 of [28]). For the statement, we call a divisor D *special* if $|K - D| \neq \emptyset$, and *non-special* otherwise.

Corollary 3.5 (Clifford's Theorem for Graphs)**.** *Let D be an effective special divisor on a graph G. Then*

$$r(D) \leq \frac{1}{2} \deg(D) \ .$$

Proof. If D is effective and special, then $K - D$ is also effective, and by Lemma 2.1 we have

$$r(D) + r(K - D) \leq r(K) = g - 1 \ .$$

On the other hand, by Riemann-Roch we have

$$r(D) - r(K - D) = \deg(D) + 1 - g \ .$$

Adding these two expressions gives $2r(D) \leq \deg(D)$ as desired. □

As pointed out in §IV.5 of [18], the interesting thing about Clifford's theorem is that for a non-special divisor D, we can compute $r(D)$ exactly as a function of $\deg(D)$ using Riemann-Roch. However, for a special divisor, $r(D)$ does not depend only on the degree. Therefore it is useful to have a non-trivial upper bound on $r(D)$, and this is what Corollary 3.5 provides.

4. The Abel-Jacobi map from a graph to its Jacobian

Let G be a graph, let $v_0 \in V(G)$ be a base point, and let k be a positive integer. In this section, we discuss the injectivity and surjectivity of the map $S_{v_0}^{(k)}$.

We leave it to the reader to verify the following elementary observations:

Lemma 4.1. 1. *$S_{v_0}^{(k)}$ is injective if and only if whenever D, D' are effective divisors of degree k with $D \sim D'$, we have $D = D'$. If $S_{v_0}^{(k)}$ is injective, then $S_{v_0}^{(k')}$ is injective for all positive integers $k' \leq k$.*

2. *$S_{v_0}^{(k)}$ is surjective if and only if every divisor of degree k is linearly equivalent to an effective divisor. If $S_{v_0}^{(k)}$ is surjective, then $S_{v_0}^{(k')}$ is surjective for all integers $k' \geq k$.*

In particular, whether or not $S_{v_0}^{(k)}$ is injective (resp. surjective) is independent of the base point v_0. We therefore write $S^{(k)}$ instead of $S_{v_0}^{(k)}$ in what follows.

4.1. **Surjectivity of the maps $S^{(k)}$.** We recall the statement of Theorem 1.7:

Theorem. *The map $S^{(k)}$ is surjective if and only if $k \geq g$.*

Proof of Theorem 1.7. This is an easy consequence of the Riemann-Roch theorem for graphs. If D is a divisor of degree $d \geq g$, then since $r(K - D) \geq -1$, Riemann-Roch implies that $r(D) \geq 0$, so that D is linearly equivalent to an effective divisor. Thus $S^{(d)}$ is surjective. (Alternatively, we can apply (RR1) directly: if $\deg(D) \geq g$, then for all $\nu \in \mathcal{N}$ we have $\deg(\nu - D) < 0$ and thus $r(\nu - D) = -1$. By (RR1) we thus have $r(D) \geq 0$.)

Conversely, (RR1) implies that $\mathcal{N} \neq \emptyset$, and therefore $S^{(g-1)}$ is not surjective. □

Remark 4.2. This result was posed as an unsolved problem on p. 179 of [2].

4.2. **The chip-firing game revisited.** As mentioned earlier, Theorems 1.9 and 1.7 are equivalent. To see this, we note the following easy lemma:

Lemma 4.3. *Two divisors D and D' on G are linearly equivalent if and only if there is a sequence of moves in the chip firing game which transforms the configuration corresponding to D into the configuration corresponding to D'.*

Proof. A sequence of moves in the chip-firing game can be encoded as the function $f \in \mathcal{M}(G)$ for which $f(v)$ is the number of times vertex v "borrows" a dollar minus the number of time it "lends" a dollar. (Note that the game is "commutative", in the sense that the order of the moves does not matter.) The ending configuration, starting from the initial configuration D and playing the moves corresponding

to f, is given by the divisor $D + \Delta(f)$. So the dollar distributions achievable from the initial configuration D are precisely the divisors linearly equivalent to D. $\square$

The equivalence between Theorem 1.9 and Theorem 1.7 is now an immediate consequence of Lemma 4.1(1), since as we have already noted, there is a winning strategy in the chip-firing game whose initial configuration corresponds to D if and only if D is linearly equivalent to an effective divisor. In particular, we have now proved Theorem 1.9.

4.3. **Injectivity of the maps $S^{(k)}$.** We recall the statement of Theorem 1.8.

Theorem. *The map $S^{(k)}$ is injective if and only if G is $(k+1)$-edge-connected.*

Proof. Suppose G is $(k+1)$-edge-connected. Choose $v_0 \in V(G)$ arbitrarily, and let $D \in \mathrm{Div}_+^k(G)$. For every non-empty $A \subseteq V(G) - \{v_0\}$, we have $\sum_{v\in A} D(v) \leq k < \sum_{v\in A} \mathrm{outdeg}_A(v)$, as $\sum_{v\in A} \mathrm{outdeg}_A(v)$ is equal to the size of the edge cut between A and $V(G) - A$. Therefore $D(v) < \mathrm{outdeg}_A(v)$ for some $v \in A$. It follows that D is v_0-reduced, so from Proposition 3.1 we deduce that no two distinct divisors in $\mathrm{Div}_+^k(G)$ are equivalent, and therefore that the map $S^{(k)}$ is injective.

Conversely, suppose G is not $(k+1)$-edge-connected. Let $C \subseteq E(G)$ be an edge cut of size $j \leq k$, and let $X \subseteq V(G)$ be one of the components of $G - C$. Let $D = \sum_{v\in X} |E_v \cap C|(v)$ and $D' = D - \Delta(\chi_X)$. Then for each $v \in V(G)$, we have

$$\begin{aligned} D'(v) &= |E_v \cap C| \cdot \chi_X(v) - \sum_{e=vw\in E_v} (\chi_X(v) - \chi_X(w)) \\ &= \begin{cases} 0 & v \in X \\ |\{e = vw \in E_v \ : \ w \in X\}| & v \notin X \ . \end{cases} \end{aligned}$$

Thus $D, D' \geq 0$, $D \sim D'$, and $D \neq D'$. It follows that the map $S^{(j)}$ is not injective, and consequently neither is $S^{(k)}$. $\square$

In particular, S is injective if and only if every edge of G is contained in a cycle.

Remark 4.4. In part (iv) of Proposition 7 in [2], the authors state that S is injective if G has vertex connectivity at least 2, and is not the graph consisting of one edge connecting two vertices. However, their proof contains an error (the map $h : V \to \mathbb{Z}/n\mathbb{Z}$ which they define need not be harmonic). In any case, Theorem 1.8 for $k = 1$ is a more precise result.

4.4. **Injectivity of the Abel-Jacobi map via circuit theory.** There is an alternate way to see that S is injective if and only if G is 2-edge-connected using the theory of electrical networks (which we refer to henceforth as *circuit theory*). We sketch the argument here; see §15 of [4] for some background on electrical networks.

Consider G as an electric circuit where the edges are resistors of resistance 1, and let $i^v_{v_0}(e)$ be the current flowing through the oriented edge e when one unit of current enters the circuit at v and exits at v_0. Let $d : C^0(G,\mathbb{R}) \to C^1(G,\mathbb{R})$ and $d^* : C^1(G,\mathbb{R}) \to C^0(G,\mathbb{R})$ be the usual operators on cochains (see §1 of [2]). By Kirchhoff's laws, $i^v_{v_0}$ is the unique element i of $C^1(G,\mathbb{R}) \cap \mathrm{Im}(d)$ for which $d^*(i) = (v) - (v_0)$. It follows from the fact that $d(C^0(G,\mathbb{Z})) = C^1(G,\mathbb{Z})$ that $i^v_{v_0} \in \mathbb{C}^1(G,\mathbb{Z})$ if and only if $(v) - (v_0) \in d^*(C^1(G,\mathbb{Z})) = (d^*d)(C^0(G,\mathbb{Z}))$, which happens if and only if $S_{v_0}(v) = 0$.

Circuit theory implies that $0 < |i^v_{v_0}(e)| \leq 1$ for every edge e which belongs to a path connecting v and v_0. In other words, the magnitude of the current flow is at most 1 everywhere in the circuit, and a nonzero amount of current must flow along every path from v to v_0.

Recall that a graph G is 2-edge-connected if and only if every edge of G is contained in a cycle. So if G is 2-edge-connected, then circuit theory implies that $|i^v_{v_0}(e)| < 1$ for every edge e belonging to a path connecting v and v_0. (Some current flows along each path from v to v_0, and there are at least two such edge-disjoint paths.) Therefore $i^v_{v_0} \notin C^1(G,\mathbb{Z})$, so $S_{v_0}(v) \neq 0$. Since $S_{v_0}(v) - S_{v_0}(v') = S_{v'}(v)$, this implies that S_{v_0} is injective.

Conversely, if an edge e' of G is not contained in any cycle, then letting v, v' denote the endpoints of e', it follows from circuit theory that

$$i^v_{v_0}(e) = \begin{cases} 1 & \text{if } e = e' \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $S_{v_0}(v) = S_{v_0}(v')$ and S_{v_0} is not injective.

Remark 4.5. A similar argument is given in §9 of [12], although the connection with the Jacobian of a finite graph is not explicitly mentioned. Yet another proof of the statement "S is injective if and only if G is 2-edge-connected" can be found in Corollary 2.3 of [23] (where the result is attributed to Hans Gerd Evertz).

The circuit theory argument actually tells us something more precise about the failure of S to be injective on a general graph G. Let $\overline{G}$ be the graph obtained by contracting every edge of G which is not part of a cycle, and let $\rho : G \to \overline{G}$ be the natural map.

Lemma 4.6. *$\rho(v_1) = \rho(v_2)$ if and only if $(v_1) \sim (v_2)$.*

Proof. $\rho(v_1) = \rho(v_2)$ if and only if there is a path from v_1 to v_2 in G, none of whose edges belong to a cycle. By circuit theory, this occurs if and only if there is a unit current flow from v_1 to v_2 which is integral along each edge. By the above discussion, this happens if and only if $(v_1) \sim (v_2)$. □

As a consequence of Lemma 4.6 and Theorem 1.8, we obtain:

Corollary 4.7. *For every graph G and every base point $v_0 \in G$, there is a commutative diagram*

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \overline{G} \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle \overline{S}} \\ \mathrm{Jac}(G) & \xrightarrow[\cong]{\rho_*} & \mathrm{Jac}(\overline{G}) \end{array}$$

in which ρ_* is an isomorphism, ρ is surjective, and $\overline{S} = \overline{S}_{\rho(v_0)}$ is injective.

Remark 4.8. (i) It is not hard to give a rigorous proof of Corollary 4.7 which does not rely on circuit theory by showing that the natural map $\rho_* : \mathrm{Div}(G) \to \mathrm{Div}(G')$ given by $\rho_*(\sum a_v(v)) = \sum a_v(\rho(v))$ sends principal divisors to principal divisors and induces a bijection $\mathrm{Jac}(G) \to \mathrm{Jac}(G')$. We leave this as an exercise for the interested reader.

(ii) Theorem 1.8 and Corollary 4.7 suggest that from the point of view of Abel-Jacobi theory, the "correct" analogue of a Riemann surface is a 2-edge-connected graph. This point of view resonates with the classification of Riemann surfaces by genus. For example, there is a unique Riemann surface of genus 0 (the Riemann sphere), and there is a unique 2-edge-connected graph of genus 0 (the graph with one vertex and no edges). Similarly, Riemann surfaces of genus 1 are classified up to isomorphism by a single complex number known as the "j-invariant", and a 2-edge-connected graph of genus 1 is isomorphic to a cycle of length $n \geq 2$, so is determined up to isomorphism by the integer n.

5. Complements

5.1. **Morphisms between graphs.** In algebraic geometry, one is usually interested not just in Riemann surfaces themselves but also in the holomorphic maps between them. The most general graph-theoretic

analogue of a holomorphic map between Riemann surfaces in the context of the present paper appears to be the notion of a *harmonic morphism*, as defined in [41]. For a non-constant harmonic morphism $f : X_1 \to X_2$, there is a graph-theoretic analogue of the classical *Riemann-Hurwitz formula* relating the canonical divisor on X_1 to the pullback of the canonical divisor on X_2. Moreover, a non-constant harmonic morphism $f : X_1 \to X_2$ induces maps $f_* : \mathrm{Jac}(X_1) \to \mathrm{Jac}(X_2)$ and $f^* : \mathrm{Jac}(X_2) \to \mathrm{Jac}(X_1)$ between the Jacobians of X_1 and X_2 in a functorial way. We will discuss these and other matters, including several characterizations of "hyperelliptic" graphs, in a subsequent paper.

5.2. **Generalizations.** There are some obvious ways in which one might attempt to generalize the results of this paper. For example:

1. We have dealt in this paper only with finite unweighted graphs, but it would be interesting to generalize our results to certain infinite graphs, as well as to weighted and/or metric graphs.

2. Can the quantity $r(D) - r(K - D)$ appearing in Theorem 1.12 be interpreted in a natural way as an Euler characteristic? In other words, is there a Serre duality theorem for graphs?

3. One could try to generalize some of the results in this paper to higher-dimensional simplicial complexes. For example, is there a higher-dimensional generalization of Theorem 1.12 analogous to the Hirzebruch-Riemann-Roch theorem in algebraic geometry?

5.3. **Other Riemann-Roch theorems.**

1. Metric graphs are closely related to "tropical curves", and in this context Mikhalkin and Zharkov have recently announced a tropical Abel-Jacobi theorem and a tropical Riemann-Roch inequality (see §5.2 of [27]). It appears, however, that their definition of $r(D)$ is different from ours (this is related to the discussion in Remark 1.13).

2. There is a Riemann-Roch formula in toric geometry having to do with lattice points and volumes of polytopes (see, e.g., §5.3 of [14]). Our Theorem 1.12 appears to be of a rather different nature.

5.4. **Connections with number theory.** The first author's original motivation for looking at the questions in this paper came from connections with number theory. We briefly discuss a few of these connections.

1. The Jacobian of a finite graph arises naturally in the branch of number theory known as arithmetic geometry. One example is the theorem of Raynaud [34] relating a proper regular semistable model $\mathcal{X}$ for

a curve X over a discrete valuation ring to the group of connected components Φ of the special fiber of the Néron model of the Jacobian of X. Although not usually stated in this way, Raynaud's result essentially says that Φ is canonically isomorphic to the Jacobian of the dual graph of the special fiber of $\mathcal{X}$. See [12, 21, 22, 23] for further details and discussion. Raynaud's theorem plays an important supporting role in a number of seminal papers in number theory (see, for example, [24] and [35]).

2. The canonical divisor K on a graph, as defined in (1.11), plays a prominent role in Zhang's refinement of Arakelov's intersection pairing on an arithmetic surface (see [43]).

3. By its definition as a "Picard group", the Jacobian of a finite graph G can be thought of as analogous to the ideal class group of a number field. In particular, the number $\kappa(G)$ of spanning trees in a graph G, which is the order of $\mathrm{Jac}(G)$, is analogous to the class number of a number field. This analogy appears explicitly in a graph-theoretic analogue (involving the Ihara zeta function of G) of the analytic class number formula for the Dedekind zeta function of a number field, see [19, p.11]. See also [20, 36, 37, 38] for further information about the Ihara zeta function of a graph.

5.5. **The chip-firing game of Björner-Lovász-Shor.** In this section, we describe some connections between our chip-firing game, as described in §1.5, and the game previously studied by Björner, Lovász, and Shor in [9]. In order to distinguish between the two, we refer to our game as the "unconstrained chip-firing game", and to the game from [9] as the "constrained chip-firing game".

The constrained chip-firing game is played as follows. Each vertex of a given (connected) graph G begins with some nonnegative amount of chips, and a move consists of choosing a vertex with at least as many chips as its degree, and having it send one chip to each of its neighbors (in which case we say that the vertex "fires"). The game terminates when no vertex is able to fire. The main results of [9] are the following two theorems:

Theorem 5.1 (Theorem 2.1 of [9])**.** *The finiteness or non-finiteness of the constrained chip-firing game, as well as the terminal configuration and the total number of moves when the game is finite, are independent of the particular moves made.*

Theorem 5.2 (Theorem 3.3 of [9])**.** *Let N be the number of chips present at any point during the constrained chip-firing game.*

(a) If $N > 2|E(G)| - |V(G)|$, the game is infinite.

(b) If $|E(G)| \leq N \leq 2|E(G)| - |V(G)|$, then there exists an initial configuration guaranteeing finite termination, and also one guaranteeing an infinite game.

(c) If $N < |E(G)|$, the game terminates in a finite number of moves.

We do not have much new to say about Theorem 5.1. However, we will show that Theorem 5.2 can be deduced from Theorem 1.9, and conversely that Theorem 5.2 implies the special case of Theorem 1.9 in which the initial configuration D satisfies $D(v) \leq \deg(v) - 1$ for all $v \in V(G)$.

The result which is needed to relate the two games is the following:

Lemma 5.3. *A winning strategy exists in the unconstrained chip-firing game with initial configuration D if and only if there is a sequence of borrowings by vertices having a negative number of dollars which transforms D into an effective divisor.*

Proof. As one direction is obvious, it suffices to show that if $D \sim E$ with $E \geq 0$, then we can get from D to an effective divisor E' via a (possibly empty) sequence of borrowings by vertices having a negative number of dollars. Since $D \sim E$, we have $E = D + \Delta(f)$ for some $f \in \mathcal{M}(G)$.

Let $E' = D + \Delta(f')$ be chosen so that:

(i) E' can be reached from D via a (possibly empty) sequence of borrowings by vertices having a negative number of dollars;
(ii) $f' \leq f$; and
(iii) $\sum_{v \in V(G)} f'(v)$ is maximal subject to conditions (i) and (ii).

We must have $E'(v) \geq 0$ for every $v \in V(G)$ such that $f'(v) < f(v)$, as otherwise the configuration $E' + \Delta(\chi_{\{v\}})$ obtained from E' by having v borrow a dollar from each of its neighbors would contradict the choice of E'. Moreover, $E'(v) \geq E(v) \geq 0$ for every $v \in V(G)$ such that $f'(v) = f(v)$. Therefore E' is effective, and the lemma holds. □

As a consequence of Lemma 5.3, we can show that the two chip firing games are related by a simple correspondence. For $D \in \mathrm{Div}(G)$, define $D^\star = K^+ - D$, where

$$K^+ = \sum_{v \in V(G)} (\deg(v) - 1)(v).$$

Explicitly, if $D = \sum a_v(v) \in \mathrm{Div}(G)$, then $D^\star = \sum a_v^\star(v)$, where $a_v^\star = \deg(v) - 1 - a_v$. Note that $a_v^\star \geq 0$ if and only if $a_v \leq \deg(v) - 1$, and that $(D^\star)^\star = D$.

Corollary 5.4. *If $D = \sum a_v(v) \in \mathrm{Div}(G)$ with $a_v \leq \deg(v) - 1$ for all $v \in V(G)$, then $|D| \neq \emptyset$ if and only if there is a legal sequence of firings in the constrained chip-firing game which starts with the configuration $D^\star$ and terminates in a finite number of moves.*

Proof. By Lemmas 4.3 and 5.3, we have $|D| \neq \emptyset$ if and only if there is a sequence of borrowings by (not necessarily distinct) vertices $v_1, \ldots, v_k$ of G that leads to a nonnegative divisor $E = \sum e_v(v)$, and such that only vertices which are in debt ever borrow. Using the definitions, this happens if and only if firing $v_1, \ldots, v_k$ in the constrained chip-firing game beginning at $D^\star$ yields a legal sequence of moves ending with a divisor $E^\star = \sum e^\star_v(v)$ having $e^\star_v \leq \deg(v) - 1$ for all $v \in V(G)$. □

With the help of Corollary 5.4, we can use Theorem 1.9 to give an alternative proof of Theorem 5.2. Indeed, suppose the constrained chip-firing game begins with a configuration $D^\star$ with $\deg(D^\star) = N$. Then $\deg(D) = 2|E(G)| - |V(G)| - N$, and by Theorem 1.9, Corollary 5.4, and the fact that $|D'| = \emptyset$ whenever $\deg(D') < 0$, we see that:

(a) If $\deg(D) < 0$, the game is infinite.

(b) If $0 \leq \deg(D) \leq |E(G)| - |V(G)| = g - 1$, then there exists an initial configuration guaranteeing finite termination, and also one guaranteeing an infinite game.

(c) If $\deg(D) > |E(G)| - |V(G)| = g - 1$, the game terminates in a finite number of moves.

This clearly implies Theorem 5.2. The same reasoning shows that Theorem 5.2 implies Theorem 1.9 in the special case where $D(v) \leq \deg(v) - 1$ for all $v \in V(G)$.

Remark 5.5. (i) Theorem 1 of [40] provides a short and elegant proof of Theorem 5.1, and can also be used to show that in the unconstrained chip-firing game with initial configuration D, every sequence of borrowings from vertices having a negative number of dollars is either infinite (if $|D| = \emptyset$) or else terminates in the same number of moves (when $|D| \neq \emptyset$). In the latter case, just as in the constrained chip-firing game, the terminal configuration is independent of the particular moves made.

(ii) If any (or equivalently, every) sequence of borrowings by vertices in debt starting with the initial configuration D terminates, then by an argument from [39] it terminates in at most $\deg^+(D)d(G)|V(G)|$ steps, where $d(G)$ denotes the diameter of G, i.e., the maximum path-distance between two vertices of G. Thus there exists an algorithm for determining whether $|D| = \emptyset$ whose running time is bounded from above by $\deg^+(D)d(G)|V(G)|$.

5.6. **Reduced divisors and critical configurations.** In [5, 6] (see also Chapter 14 of [16]), Biggs studies the *critical group* of a graph, which he defines in terms of a certain chip-firing game played on the vertices of the graph. One of Biggs' results is that the critical group is isomorphic to $\mathrm{Jac}(G)$. In this section, we describe a one-to-one correspondence between elements of Biggs' critical group and v_0-reduced divisors, as defined in §3.1. In order to do this, we first need to translate Biggs' definitions into the language of divisors.

Let $v_0 \in V(G)$ be a fixed base vertex, and let $v_1, v_2, \ldots, v_{n-1}$ be an ordering of the vertices in $V(G) - \{v_0\}$, where $n = |V(G)|$. We say that a divisor D is v_0-*critical with respect to the ordering* $v_1, v_2, \ldots, v_{n-1}$ if for every $v \in V(G) - \{v_0\}$ we have $0 \leq D(v) \leq \deg(v) - 1$, and if for every $1 \leq k \leq n-1$ we have $D_k(v) \geq 0$, where

$$D_k = D - \sum_{i=0}^{k} \Delta(\chi_{\{v_i\}}).$$

We say that a divisor D is v_0-*critical* if it is v_0-critical with respect to some ordering of $V(G) - \{v_0\}$.

We remark on some technical differences between the above definition and the definition given in [5]. In [5], only configurations for which the total amount of money is zero are considered. Also, the definition of a critical configuration given in [5], when translated directly into the language of divisors, would appear to be slightly different from ours; however, the two definitions are in fact equivalent by Lemma 2.6 of [5].

It follows from the results of [5] and [6] that given $v_0 \in V(G)$, every equivalence class of $\mathrm{Div}(G)$ contains a unique v_0-critical divisor. This observation suggests a relationship between v_0-reduced and v_0-critical divisors. In the following lemma, we show that in fact there exists a natural bijection between the two.

Lemma 5.6. *A divisor D is v_0-reduced if and only if the divisor $D^\star = K^+ - D$ is v_0-critical.*

Proof. Let $n = |V(G)|$. Suppose that D is v_0-reduced, and define $v_1, v_2, \ldots, v_{n-1}$ as in the proof of Theorem 3.3. We claim that $D^\star$ is v_0-critical with respect to this ordering of $V(G) - \{v_0\}$.

Write $D^\star_k$ for $(D^\star)_k = K^+ - D - \Delta(\chi_{B_k})$, where $B_k = \{v_0, v_1, \ldots, v_k\}$. Let $v \in V(G) - \{v_0\}$. We have $0 \leq D(v) < \mathrm{outdeg}_{\{v\}}(v) = \deg(v)$ and therefore $0 \leq D^\star(v) < \deg(v)$. It remains to prove that $0 \leq D^\star_k(v)$ for every $1 \leq k \leq n-1$.

If $v \notin B_k$, then $D_k^\star(v) \geq D^\star(v) \geq 0$. Otherwise $v = v_l$ for some $0 < l \leq k$, and

$$\begin{aligned} D_k^\star(v_l) \geq D_l^\star(v_l) &= \deg(v_l) - 1 - D(v_l) - \mathrm{outdeg}_{B_l}(v_l) \\ &= \big(\deg(v_l) - \mathrm{outdeg}_{B_l}(v_l)\big) - D(v_l) - 1 \\ &= \mathrm{outdeg}_{V(G)-B_{l-1}}(v_l) - D(v_l) - 1 \\ &\geq 0 \ , \end{aligned}$$

where the last inequality follows from the definition of v_l. (Here we have used the fact that if $A \subseteq V(G)$, then $\mathrm{outdeg}_A(v) + \mathrm{outdeg}_{V(G)-A}(v) = \deg(v)$ for all $v \in V(G)$, and if $v \in A$, then $\mathrm{outdeg}_A(v) = \mathrm{outdeg}_{A-\{v\}}(v)$.) It follows that $D^\star$ is v_0-critical with respect to the given order, as desired.

Now suppose $D^\star$ is v_0-critical with respect to the ordering $v_1, v_2, \ldots, v_{n-1}$. Consider a non-empty subset $A \subseteq V(G) - \{v_0\}$, and let v_l be the vertex in A having the smallest index. We have

$$0 \leq D_l^\star(v_l) = \mathrm{outdeg}_{V(G)-B_{l-1}}(v_l) - D(v_l) - 1 \ ,$$

where B_{l-1} is defined as above. Moreover, $B_{l-1} \cap A = \emptyset$, and therefore

$$D(v_l) < \mathrm{outdeg}_{V(G)-B_{l-1}}(v_l) \leq \mathrm{outdeg}_A(v_l) \ .$$

As $A \subseteq V(G) - \{v_0\}$ was arbitrary, we conclude that D is v_0-reduced. □

Remark 5.7. Lemma 5.6 explains some of the parallels found in the literature between certain results concerning G-parking functions and critical configurations. As two examples, we mention:

(i) The construction of explicit bijections between G-parking functions and spanning trees from [11], and between critical configurations and spanning trees in [7].
(ii) The relationship between G-parking functions and the Tutte polynomial, as described in [32], and between critical configurations and the Tutte polynomial, as described in [26] and [6].

Appendix A. Riemann surfaces and their Jacobians

The theory of Riemann surfaces and their Jacobians is one of the major accomplishments of 19th century mathematics, and it continues to this day to have significant applications. We cannot hope to give the reader a complete overview of this vast subject, so we will just touch on a few of the highlights of the theory in order to draw out the connections with graph theory. We recommend [28] as a good

introduction to the theory of Riemann surfaces and their Jacobians; see also [1, 13, 17, 18, 29, 30].

A (compact) *Riemann surface* X is a one-dimensional connected complex manifold, i.e., a two-dimensional connected compact real manifold endowed with a maximal atlas $\{U_\alpha, z_\alpha\}$ for which the transition functions

$$f_{\alpha\beta} = z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\alpha \cap U_\beta)$$

are holomorphic whenever $U_\alpha \cap U_\beta \neq \emptyset$.

The simplest example of a Riemann surface is the *Riemann sphere* $\mathbb{C} \cup \{\infty\}$.

Since a Riemann surface looks locally like an open subset of $\mathbb{C}$, there is a natural notion of what is means for a function $f : X \to \mathbb{C}$ (resp. $f : X \to \mathbb{C} \cup \{\infty\}$) to be *holomorphic* (resp. meromorphic): we say that f is holomorphic (resp. meromorphic) if $f \circ z^{-1}$ is holomorphic (resp. meromorphic) for every coordinate chart (U, z).

A 1-*form* ω on a Riemann surface X is a collection of 1-forms $\omega_x dx + \omega_y dy$ on each coordinate chart (U, z) (where $z = x + iy$) satisfying suitable compatibility relations on overlapping charts. A 1-form is *holomorphic* if ω_x and ω_y are holomorphic and $\omega_y = i\omega_x$. Locally, every holomorphic 1-form is equal to $f(z)dz$ with f a holomorphic function. Finally, a 1-form is *meromorphic* if it is holomorphic outside a finite set of points and can be represented locally as $f(z)dz$ with f a meromorphic function.

Riemann surfaces are classified by a nonnegative integer g called the *genus*. There are several equivalent characterizations of the genus of a Riemann surface; for example, $2g$ is the topological genus of X, i.e., $\dim_{\mathbb{R}} H_1(X, \mathbb{R})$, and g is the complex dimension of the space of holomorphic 1-forms on X. A Riemann surface has genus 0 if and only if it is isomorphic to the Riemann sphere.

Let $\mathrm{Div}(X)$ be the free abelian group on the set of vertices of X; elements of $\mathrm{Div}(X)$ are called *divisors* on X and are usually written as $\sum_{P \in X} a_P(P)$, where each a_P is an integer and all but finitely many of the a_P's are zero. A divisor $E \in \mathrm{Div}(X)$ is called *effective* if $E \geq 0$.

There is a natural *degree function* $\deg : \mathrm{Div}(X) \to \mathbb{Z}$ given for $D = \sum a_P(P)$ by

$$\deg(D) = \sum_{P \in X} a_P \ .$$

If $\mathcal{M}(X)$ denotes the space of meromorphic functions on X, then for every nonzero $f \in \mathcal{M}(X)$ and every $P \in X$, one can define, using

local coordinates, the *order of vanishing* $\mathrm{ord}_P(f)$ of f at P. For all but finitely many $P \in X$, one has $\mathrm{ord}_P(f) = 0$. The *divisor of* f is then defined to be

$$\mathrm{div}(f) = \sum_{P \in X} \mathrm{ord}_P(f)(P) \ . \tag{A.1}$$

The divisor of a nonzero meromorphic function f is called a *principal divisor*. A fundamental fact about Riemann surfaces is that $\deg(\mathrm{div}(f)) = 0$, which means that f has the same number of zeros as poles (counting multiplicities). Therefore $\mathrm{Prin}(X)$ (the set of all principal divisors) is a subgroup of the group $\mathrm{Div}^0(X)$ of divisors of degree zero.

The Jacobian $\mathrm{Jac}(X)$ of X (also denoted $\mathrm{Pic}^0(X)$) is defined to be the quotient group

$$\mathrm{Jac}(X) = \frac{\mathrm{Div}^0(X)}{\mathrm{Prin}(X)} \ . \tag{A.2}$$

The abelian group $\mathrm{Jac}(X)$ is naturally endowed with the structure of a (projective) compact complex manifold of dimension g, i.e., $\mathrm{Jac}(X)$ is an *abelian variety*.

Two divisors D, D' on X are called *linearly equivalent* if their difference is a principal divisor. Thus $\mathrm{Jac}(X)$ classifies the degree zero divisors on X modulo linear equivalence.

If we fix a base point $P_0 \in \mathrm{Jac}(X)$, we can define the *Abel-Jacobi map* $S_{P_0} : X \to \mathrm{Jac}(X)$ by the formula

$$S_{P_0}(P) = [(P) - (P_0)] \ , \tag{A.3}$$

where $[D]$ denotes the class in $\mathrm{Jac}(X)$ of $D \in \mathrm{Div}^0(X)$. We write S instead of S_{P_0} when the base point P_0 is understood.

We can also define, for each $k \geq 1$, the map $S^{(k)}_{P_0} : \mathrm{Div}^k_+(X) \to \mathrm{Jac}(X)$ by

$$S^{(k)}_{P_0}((P_1) + \cdots + (P_k)) = S_{P_0}(P_1) + S_{P_0}(P_2) + \cdots + S_{P_0}(P_k) \ ,$$

where $\mathrm{Div}^k_+(X)$ denotes the set of effective divisors of degree k on X.

The map S_{v_0} can be characterized by the following universal property: If φ is a holomorphic map from X to an abelian variety A taking P_0 to 0, then there is a unique homomorphism $\psi : \mathrm{Jac}(X) \to A$ such that $\varphi = \psi \circ S_{P_0}$.

A classical result about the maps $S^{(k)}$ is the following:

Theorem A.4. *$S^{(k)}$ is surjective if and only if $k \geq g$.*

The surjectivity of $S^{(g)}$ is usually referred to as *Jacobi's inversion theorem*; it is equivalent to the statement that every divisor of degree at least g on X is linearly equivalent to an effective divisor.

Another classical fact is:

Theorem A.5. *The Abel-Jacobi map S is injective if and only if $g \geq 1$.*

Let D be a divisor on X. The *linear system associated to D* is defined to be the set $|D|$ of all effective divisors linearly equivalent to D:

$$|D| = \{E \in \mathrm{Div}(X) \ : \ E \geq 0, \ E \sim D\} \ .$$

The *dimension* $r(D)$ of the linear system $|D|$ is defined to be one less than the dimension of $L(D)$, where

$$L(D) = \{f \in \mathcal{M}(X) \ : \ \mathrm{div}(f) \geq -D\}$$

is the finite-dimensional $\mathbb{C}$-vector space consisting of all meromorphic functions for which $\mathrm{div}(f) + D$ is effective. There is a natural identification

$$|D| = (L(D) - \{0\}) / \mathbb{C}^*$$

of $|D|$ with the projectivization of $L(D)$. It is easy to see that $r(D)$ depends only on the linear equivalence class of D.

Remark A.6. In the graph-theoretic setting, the analogue of $L(D)$ is no longer a vector space. Therefore it is useful to have a more intrinsic characterization of the quantity $r(D)$ in terms of $|D|$ only. Such a characterization is in fact well-known (see, e.g., p.250 of [17] or §III.8.15 of [13]): $r(D) \geq -1$ for all D, and for each $s \geq 0$ we have $r(D) \geq s$ if and only if $|D - E| \neq \emptyset$ for all effective divisors E of degree s.

Given a nonzero meromorphic 1-form ω on X, one can define (using local coordinates) the order of vanishing of ω at a point $P \in X$, and the divisor $\mathrm{div}(\omega)$ of ω is then defined as in (A.1). The degree of $\mathrm{div}(\omega)$ is $2g - 2$ for every ω, and if ω, ω' are both nonzero meromorphic 1-forms on X, the quotient ω/ω' is a nonzero meromorphic function on X, and thus $\mathrm{div}(\omega)$ and $\mathrm{div}(\omega')$ are linearly equivalent.

The *canonical divisor class* K_X on X is defined to be the linear equivalence class of $\mathrm{div}(\omega)$ for any nonzero meromorphic 1-form ω.

The following result, known as the *Riemann-Roch theorem*, is widely regarded as the single most important result in the theory of Riemann surfaces.

Theorem A.7 (Riemann-Roch)**.** *Let X be a Riemann surface with canonical divisor class K, and let D be a divisor on X. Then*

$$r(D) - r(K - D) = \deg(D) + 1 - g \ .$$

The importance of Theorem A.7 stems from the large number of applications which it has; see, e.g., Chapters VI and VII of [28] and Chapter IV of [18].

Finally, we discuss *Abel's theorem*, which gives an alternative characterization of $\mathrm{Jac}(X)$ and the Abel-Jacobi map $S_{P_0} : X \to \mathrm{Jac}(X)$.

Choose a base point $P_0 \in X$, and let $\Omega^1(X)$ denote the space of holomorphic 1-forms on X. Every (integral) homology class $\gamma \in H_1(X, \mathbb{Z})$ defines an element $\int_\gamma$ of the dual space $\Omega^1(X)^*$ via integration:

$$\int_\gamma : \omega \mapsto \int_\gamma \omega \in \mathbb{C} \ .$$

A linear functional $\lambda : \Omega^1(X) \to \mathbb{C}$ is called a *period* if it is of the form $\int_\gamma$ for some $\gamma \in H_1(X, \mathbb{Z})$. We let Λ denote the set of periods; it is a lattice in $\Omega^1(X)^*$.

For each point $P \in X$, choose a path γ_P in X from P_0 to P, and define $A_{P_0} : X \to \Omega^1(X)^*/\Lambda$ by sending P to class of the linear functional $\int_{\gamma_P}$ given by integration along γ_P. This is well-defined, since if γ'_P is another path from P_0 to P, then the 1-chain $\gamma_P - \gamma'_P$ is closed and therefore defines an integral homology class.

We can extend the map A_{P_0} by linearity to a homomorphism from $\mathrm{Div}(X)$ to $\Omega^1(X)^*/\Lambda$. Restricting to $\mathrm{Div}^0(X)$ gives a canonical map $A : \mathrm{Div}^0(X) \to \Omega^1(X)^*/\Lambda$ which does not depend on the choice of base point P_0.

Theorem A.8 (Abel's Theorem)**.** *The map A is surjective, and its kernel is precisely* $\mathrm{Prin}(X)$*. Therefore A induces an isomorphism of* $\mathrm{Jac}(X)$ *onto $\Omega^1(X)^*/\Lambda$. Moreover, we have $A_{P_0} = A \circ S_{P_0}$, i.e., A_{P_0} coincides with the Abel-Jacobi map S_{P_0} under the identification of* $\mathrm{Jac}(X)$ *and $\Omega^1(X)^*/\Lambda$ furnished by A.*

In particular, if D is a divisor of degree zero on X, then D is the divisor of a meromorphic function on X if and only if $A(D) = 0$.

Appendix B. Abel's theorem for graphs

For the sake of completeness, we recall from [2] a graph-theoretic analogue of Abel's theorem (Theorem A.8). See also [31] and §28-29 of [4] for further details.

Choose an orientation of the graph G, i.e., for each edge e pick a vertex e_+ incident to e, and let e_- be the other vertex incident to e. Let $C^0(G,\mathbb{R})$ be the $\mathbb{R}$-vector space consisting of all functions $f : V(G) \to \mathbb{R}$. Inside this space, we have the lattice $C^0(G,\mathbb{Z})$ consisting of the integer valued functions. Similarly, we can consider the space $C^1(G,\mathbb{R})$ of all functions $h : E(G) \to \mathbb{R}$ and the corresponding lattice $C^1(G,\mathbb{Z})$. We equip $C^0(G,\mathbb{R})$ and $C^1(G,\mathbb{R})$ with the inner products given by

$$\langle f_1, f_2\rangle = \sum_{v\in V(G)} f_1(v)f_2(v) \tag{B.1}$$

and

$$\langle h_1, h_2\rangle = \sum_{e\in E(G)} h_1(e)h_2(e) \ . \tag{B.2}$$

Define the *exterior differential* $d : C^0(G,\mathbb{R}) \to C^1(G,\mathbb{R})$ by the formula

$$df(e) = f(e_+) - f(e_-) \ .$$

The adjoint $d^* : C^1(G,\mathbb{R}) \to C^0(G,\mathbb{R})$ of d with respect to the inner products (B.1) and (B.2) is given by

$$(d^*h)(v) = \sum_{\substack{e\in E(G)\\ e_+=v}} h(e) - \sum_{\substack{e\in E(G)\\ e_-=v}} h(e) \ .$$

It is easily checked that $\Delta = d^*d : C^0(G,\mathbb{R}) \to \mathcal{C}^0(G,\mathbb{R})$ is independent of the choice of orientation, and can be identified with the Laplacian operator on G, i.e.:

$$(d^*df)(v) = \deg(v)f(v) - \sum_{e=wv\in E_v} f(w) \ .$$

There is an orthogonal decomposition

$$C^1(G,\mathbb{R}) = \mathrm{Ker}(d^*) \oplus \mathrm{Im}(d) \ ,$$

where $\mathrm{Ker}(d^*)$ is the *flow space* (or *cycle space*) and $\mathrm{Im}(d)$ is the *cut space* (or *potential space*).

The *lattice of integral flows* is defined to be $\Lambda^1(G) = \mathrm{Ker}(d^*) \cap C^1(G,\mathbb{Z})$, and the *lattice of integral cuts* is defined to be $N^1(G) = \mathrm{Im}(d) \cap C^1(G,\mathbb{Z})$.

For a lattice Λ in a Euclidean inner product space V, the *dual lattice* $\Lambda^\#$ is defined to be

$$\Lambda^\# = \{x \in V \ : \ \langle x,\lambda\rangle \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\} \ .$$

A lattice Λ is called *integral* if $\langle \lambda, \mu \rangle \in \mathbb{Z}$ for all $\lambda, \mu \in \Lambda$; this is equivalent to requiring that $\Lambda \subseteq \Lambda^{\#}$. Clearly $\Lambda^1(G)$ and $N^1(G)$ are integral lattices.

In the statement of the following theorem, the dual of $\Lambda^1(G)$ (resp. $\mathcal{N}^1(G)$) is defined with respect to the ambient space $\mathrm{Ker}(d^*)$ (resp. $\mathrm{Im}(d)$).

Theorem B.3. *The groups* $C^1(G,\mathbb{Z})/(\Lambda^1(G) \oplus N^1(G)), \Lambda^1(G)^{\#}/\Lambda^1(G)$, *and* $N^1(G)^{\#}/N^1(G)$ *are all isomorphic.*

Choose a base vertex $v_0 \in G$. One can describe a map $A_{v_0} : G \to J(G) := \Lambda^1(G)^{\#}/\Lambda^1(G)$ as follows. For any $v \in V(G)$, choose a path γ from v_0 to v, which may be identified in the obvious way with an element of $C^1(G,\mathbb{Z})$. If γ' is any other path from v_0 to v, then $\gamma - \gamma' \in \Lambda^1(G)$. Since $\langle \gamma, \lambda \rangle \in \mathbb{Z}$ for every $\lambda \in \Lambda^1(G)$, γ determines an element A_γ of $\Lambda^1(G)^{\#}$. We define $A_{v_0}(v)$ to be the class of A_γ in $\Lambda^1(G)^{\#}/\Lambda^1(G)$; this is independent of the choice of γ.

We can extend the map A_{v_0} by linearity to a homomorphism from $\mathrm{Div}(G)$ to $\Lambda^1(G)^{\#}/\Lambda^1(G)$. Restricting to $\mathrm{Div}^0(G)$ gives a canonical map $A : \mathrm{Div}^0(G) \to J(G)$ which does not depend on the choice of base point v_0.

Theorem B.4 (Abel's Theorem for Graphs)**.** *The map A is surjective, and its kernel is precisely* $\mathrm{Prin}(G)$. *Therefore A induces an isomorphism of* $\mathrm{Jac}(G)$ *onto* $\Lambda^1(G)^{\#}/\Lambda^1(G)$. *Moreover, we have* $A_{v_0} = A \circ S_{v_0}$, *i.e.,* A_{v_0} *coincides with the Abel-Jacobi map* S_{v_0} *defined by (1.6) under the identification of* $\mathrm{Jac}(G)$ *and* $J(G)$ *furnished by* A.

Consequently, if D is a divisor of degree zero on G, then D is principal if and only if $A(D) = 0$. For proofs of Theorems B.3 and B.4, see [2] and §24-29 of [4].

Remark B.5. The lattices $\Lambda^1(G)$ and $N^1(G)$ have a number of interesting combinatorial properties. For example, it is shown in Propositions 1 and 2 of [2] that $\Lambda^1(G)$ is even if and only if G is bipartite, and $N^1(G)$ is even if and only if G is Eulerian. Moreover, the length of the shortest nonzero vector in $\Lambda^1(G)$ is the girth of G, and the length of the shortest nonzero vector in $N^1(G)$ is the edge connectivity of G. And of course, it follows from Theorem B.3 that both $|\Lambda^1(G)^{\#}/\Lambda^1(G)|$ and $|N^1(G)^{\#}/N^1(G)|$ are equal to the number of spanning trees in G.

References

[1] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften*

[Fundamental Principles of Mathematical Sciences]. Springer-Verlag, New York, 1985.

[2] R. Bacher, P. de la Harpe, and T. Nagnibeda. The lattice of integral flows and the lattice of integral cuts on a finite graph. *Bull. Soc. Math. France*, 125(2):167–198, 1997.

[3] N. Biggs. *Algebraic graph theory*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 1993.

[4] N. Biggs. Algebraic potential theory on graphs. *Bull. London Math. Soc.*, 29(6):641–682, 1997.

[5] N. Biggs. Chip-firing and the critical group of a graph. *J. Algebraic Combin.*, 9(1):25–45, 1999.

[6] N. Biggs. The Tutte polynomial as a growth function. *J. Algebraic Combin.*, 10(2):115–133, 1999.

[7] N. Biggs and P. Winkler. Chip-firing and the chromatic polynomial. preprint, 9 pages, 1997.

[8] A. Björner and L. Lovász. Chip-firing games on directed graphs. *J. Algebraic Combin.*, 1(4):305–328, 1992.

[9] A. Björner, L. Lovász, and P. W. Shor. Chip-firing games on graphs. *European J. Combin.*, 12(4):283–291, 1991.

[10] B. Bollobás. *Modern graph theory*, volume 184 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[11] D. Chebikin and P. Pylyavskyy. A family of bijections between G-parking functions and spanning trees. *J. Combin. Theory Ser. A*, 110(1):31–41, 2005.

[12] S. J. Edixhoven. On Néron models, divisors and modular curves. *J. Ramanujan Math. Soc.*, 13(2):157–194, 1998.

[13] H. M. Farkas and I. Kra. *Riemann surfaces*, volume 71 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1992.

[14] W. Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[15] S. Gaubert. Two lectures on max-plus algebra. preprint. Available at `http://citeseer.ist.psu.edu/gaubert98two.html`, 58 pages, 1998.

[16] C. Godsil and G. Royle. *Algebraic graph theory*, volume 207 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[17] P. Griffiths and J. Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[18] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[19] M. Horton, H. Stark, and A. Terras. What are zeta functions of graphs and what are they good for? In *Quantum graphs and their applications*, volume 415 of *Contemp. Math.*, pages 173–189. Amer. Math. Soc., Providence, RI, 2006.

[20] M. Kotani and T. Sunada. Zeta functions of finite graphs. *J. Math. Sci. Univ. Tokyo*, 7(1):7–25, 2000.

[21] D. J. Lorenzini. Arithmetical graphs. *Math. Ann.*, 285(3):481–501, 1989.

[22] D. J. Lorenzini. A finite group attached to the Laplacian of a graph. *Discrete Math.*, 91(3):277–282, 1991.

[23] D. J. Lorenzini. Arithmetical properties of Laplacians of graphs. *Linear and Multilinear Algebra*, 47(4):281–306, 2000.

[24] B. Mazur. Modular curves and the Eisenstein ideal. *Inst. Hautes Études Sci. Publ. Math.*, (47):33–186 (1978), 1977.

[25] C. Merino. The chip-firing game. *Discrete Math.*, 302(1-3):188–210, 2005.

[26] C. Merino López. Chip firing and the Tutte polynomial. *Ann. Comb.*, 1(3):253–259, 1997.

[27] G. Mikhalkin. Tropical geometry and its applications. preprint. Available at `http://www.math.toronto.edu/mikha/icm.pdf`, 25 pages, 2006.

[28] R. Miranda. *Algebraic curves and Riemann surfaces*, volume 5 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1995.

[29] D. Mumford. *Curves and their Jacobians*. The University of Michigan Press, Ann Arbor, Mich., 1975.

[30] V. Kumar Murty. *Introduction to abelian varieties*, volume 3 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 1993.

[31] T. Nagnibeda. The Jacobian of a finite graph. In *Harmonic functions on trees and buildings (New York, 1995)*, volume 206 of *Contemp. Math.*, pages 149–151. Amer. Math. Soc., Providence, RI, 1997.

[32] J. Plautz and R. Calderer. G-parking functions and the Tutte polynomial. preprint. Available at `http://www.math.umn.edu/%7Ereiner/REU/PlautzReport.ps`, 3 pages, 2003.

[33] A. Postnikov and B. Shapiro. Trees, parking functions, syzygies, and deformations of monomial ideals. *Trans. Amer. Math. Soc.*, 356(8):3109–3142 (electronic), 2004.

[34] M. Raynaud. Spécialisation du foncteur de Picard. *Inst. Hautes Études Sci. Publ. Math.*, (38):27–76, 1970.

[35] K. A. Ribet. On modular representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ arising from modular forms. *Invent. Math.*, 100(2):431–476, 1990.

[36] H. M. Stark and A. A. Terras. Zeta functions of finite graphs and coverings. *Adv. Math.*, 121(1):124–165, 1996.

[37] H. M. Stark and A. A. Terras. Zeta functions of finite graphs and coverings. II. *Adv. Math.*, 154(1):132–195, 2000.

[38] H. M. Stark and A. A. Terras. Zeta functions of finite graphs and coverings. III. *Adv. Math.*, 208(1):467–489, 2007.

[39] G. Tardos. Polynomial bound for a chip firing game on graphs. *SIAM J. Discrete Math.*, 1(3):397–398, 1988.

[40] M. Thorup. Firing games. preprint. Available at `http://citeseer.ist.psu.edu/thorup96firing.html`, 2 pages, 1996.

[41] H. Urakawa. A discrete analogue of the harmonic morphism and Green kernel comparison theorems. *Glasg. Math. J.*, 42(3):319–334, 2000.

[42] J. van den Heuvel. Algorithmic aspects of a chip-firing game. *Combin. Probab. Comput.*, 10(6):505–529, 2001.

[43] S. Zhang. Admissible pairing on a curve. *Invent. Math.*, 112(1):171–193, 1993.

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