PRIMER FOR THE ALGEBRAIC GEOMETRY OF SANDPILES

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Abstract. This is a draft of a primer on the algebraic geometry of the Abelian Sandpile Model. version: December 7, 2010.

1. Introduction

[WARNING: This introduction is woefully out of date. The other sections of the primer are getting close to their final form now, although they need to be proofread.] This is a primer on the algebraic geometry of sandpiles based on lectures given in an undergraduate Topics in Algebra course at Reed College in the fall of 2008. It is assumed that the reader has no background in algebraic geometry. The second section of these notes gives an introduction to the Abelian Sandpile Model. What might be novel there is the treatment of burning configurations for directed multigraphs, but the main idea is due to [20]. Section 3 is a summary of the theory of lattice ideals as needed for the sequel. The first paper on the algebraic geometry of sandpiles of which we are aware is *Polynomial ideals for sandpiles and their Gröbner bases*, by Cori, Rossin, and Salvy [5]. That paper defines the toppling ideal of a undirected graph and computes a Gröbner basis for the ideal with respect to a certain natural monomial ordering. Section 4 extends their work, putting it in the context of lattice ideals and generalizing the Gröbner basis result to the case of directed multigraphs. It turns out that any lattice ideal whose zero set is finite is the lattice ideal corresponding to some directed multigraph. Section 6 gives an explicit description of the zero set of the toppling ideal. It is a generic orbit of a faithful representation of the sandpile group of the graph. The affine Hilbert function of the toppling ideal is defined in terms of the sandpile group. Matthew Baker et al. proved a Riemann-Roch theorem for undirected graphs. Using their language, in section 7, we see that the minimal free resolution of the homogeneous toppling ideal is graded by divisors on the graph modulo linear equivalence. The Betti numbers are determined by the simplicial homology of complexes forming the supports of complete linear systems on the graph. Finally, in Section 8, we completely characterize directed multigraphs whose homogeneous toppling ideals are complete intersection ideals. Further, we give a method of constructing directed multigraphs whose homogeneous toppling ideals are arithmetically Gorenstein.

The reader may be interested in `www.reed.edu/~davidp/sand`. Among other things, this site has links to programs that are useful in doing sandpile calculations. One of these is a package for the mathematical software system Sage [21]. The online manual for the package contains an introduction to sandpile along with examples done within Sage. Following the Google Summer of Code link will take the reader

Date: September 2, 2009.

to programs being developed to visualize and analyze the Abelian Sandpile Model.

ACKNOWLEDGMENTS. We would like to express our thanks to students who participated in the Topics in Algebra course at Reed College in fall 2008. We would also like to acknowledge the contribution of Luis David Garcia. He suggested that we look at [16] in the context of sandpile ideals, leading us to Theorems 8.10 and 8.12.

2. SANDPILES

In this section we summarize the theory of sandpile groups. For a thorough introduction to the subject, the reader is referred to [13].

2.1. **Graph theory.** Let $G = (V, E)$ be a directed multigraph with a finite sets of vertices V and directed edges E. Thus, $E \subset V \times V$. If $e = (u, v) \in E$, we write $e^- := u$ and $e^+ := v$ for the *tail* and *head* of e, respectively. If $e^- = e^+$, the edge is a *loop*. These are allowed but do not add much to the theory. By "multigraph" we will mean that there is a *weight function*:

$$\mathrm{wt}\colon V \times V \to \mathbb{N}$$

such that $w(u, v) > 0$ if and only if $(u, v) \in E$. One may think of the weight of an edge $e = (u, v)$ as $\mathrm{wt}(e)$ edges connecting u to v. For $v \in V$,

$$\begin{aligned} \mathrm{outdeg}(v) &:= \sum_{e\in E:e^-=v} \mathrm{wt}(e) \\ \mathrm{indeg}(v) &:= \sum_{e\in E:e^+=v} \mathrm{wt}(e). \end{aligned}$$

The graph G is *undirected* if $\mathrm{wt}(u, v) = \mathrm{wt}(v, u)$ for all $u, v \in V$, and it is *unweighted* if the weights of all of its edges are 1. If G is undirected, we use the notation $\deg(v) := \mathrm{outdeg}(v) = \mathrm{indeg}(v)$.

A vertex u is *accessible* from a vertex v if there is a directed path beginning at u and ending at v. A vertex s is *globally accessible* if it is accessible from all vertices of G.

Definition 2.1. A *sandpile graph* is a triple (V, E, s) consisting of a finite, directed multigraph (V, E) with a globally accessible vertex s. The vertex s is called the *sink* of the sandpile graph. The nonsink vertices are denoted $\widetilde{V} := V \setminus \{s\}$.

If $G = (V, E, s)$ is a sandpile graph, we will also refer to the graph (V, E) as G. Note that the sink of a sandpile graph need not have outdegree zero; however for most of what we say, one could safely remove outgoing edges from the sink without changing the theory. We say that s is an *absolute sink* if it is both globally accessible and has outdegree 0.

Example 2.2. Figure 1 depicts a sandpile graph G. Edges (v_1, v_2), (v_2, v_1), (v_2, s), (v_3, s), and (s, v_3) are directed edges with weights $1, 2, 1, 1, 5$, respectively; $\{v_1, v_3\}$ is an undirected edge of weight 3; and $\{v_2, v_3\}$ is an undirected, unweighted edge. Although, s is the sink of the sandpile graph, $\mathrm{outdeg}(s) = 5$.

For any finite set X, let

$$\mathbb{Z}X = \{\textstyle\sum_{x\in X} a_x\, x : a_x \in \mathbb{Z} \text{ for all } x \in X\}$$

be the free Abelian group on X. Restricting to nonnegative coefficients gives $\mathbb{N}X$.

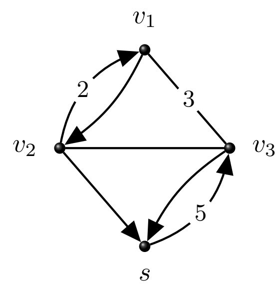


FIGURE 1. Sandpile graph G with sink s.

Notation 2.3. For $a, b \in \mathbb{Z}X$, we define $\deg(a) = \sum_{x \in X} a_x$ and $a \geq b$ if $a_x \geq b_x$ for all $x \in X$. We say a is *nonnegative* if $a \geq 0$. The *support* of a is

$$\operatorname{supp}(a) = \{x \in X : a_x \neq 0\}.$$

Similar notation is used for integer vectors.

Let $G = (V, E, s)$ be a sandpile graph.

Definition 2.4. The *(full) Laplacian* of G is the mapping of groups $\Delta\colon \mathbb{Z}V \to \mathbb{Z}V$ given on vertices v by

$$\Delta(v) := \operatorname{outdeg}(v) - \sum_{u \in V} \operatorname{wt}(v, u)\, u.$$

The *reduced Laplacian* of G is the mapping of groups $\widetilde{\Delta}\colon \mathbb{Z}\widetilde{V} \to \mathbb{Z}\widetilde{V}$ given on nonsink vertices v by

$$\widetilde{\Delta}(v) := \operatorname{outdeg}(v) - \sum_{u \in \widetilde{V}} \operatorname{wt}(v, u)\, u,$$

summing this time only over $\widetilde{V}$.

The Laplacian just defined is dual to the Laplacian one often sees in the literature. Define $L\colon \mathbb{Z}^V \to \mathbb{Z}^V$ by

$$L\phi(v) := \sum_{u \in V} \operatorname{wt}(v, u)(\phi(v) - \phi(u))$$

for a function $\phi \in \mathbb{Z}^V$ and vertex v. Let $D = \operatorname{diag}(\operatorname{outdeg}(v_1), \ldots, \operatorname{outdeg}(v_{n+1}))$, and let A be the *adjacency matrix*, A, given by $A_{ij} = \operatorname{wt}(v_i, v_j)$. Fixing an ordering $v_1, \ldots, v_{n+1}$ of the vertices, identifies $\mathbb{Z}^V$ with $\mathbb{Z}^{n+1}$ and identifies L with the $(n+1) \times (n+1)$ matrix

$$L = D - A.$$

The matrix for our Laplacian Δ of G is the transpose of L.

A *spanning tree directed into s* is a subgraph T of G with the property that (1) T contains all of the vertices of G, (2) the weight of each edge in T is the same as its weight as an edge of G, (3) for each vertex, there is a directed path in T to s, and (4) for each vertex $v \neq s$, there is exactly one edge of T whose tail is v. If T is a spanning tree directed into s, then its weight, denoted $\operatorname{wt}(T)$, is the product of the weights of its edges.

Theorem 2.5 (Matrix-Tree)**.** *The determinant of the reduced Laplacian of G is the sum of the weights of all its directed spanning trees into the sink.*

2.2. **The Sandpile Group.** Let $G = (V, E, s)$ be a sandpile graph with nonsink vertices $\widetilde{V}$.

Definition 2.6. A *(sandpile) configuration* on G is an element of $\mathbb{Z}\widetilde{V}$. A configuration $c = \sum_{v\in\widetilde{V}} c_v v$ is *stable* at a vertex $v \in \widetilde{V}$ if $c_v < \text{outdeg}(v)$. Otherwise, it is *unstable*. A configuration is *stable* if it is stable at each $v \in \widetilde{V}$.

As the name suggests, we think of a configuration c as a pile of sand on the nonsink vertices of G having c_v grains of sand at vertex v. Sand can be redistributed on the graph by vertex *firings (or topplings)*. Firing $v \in \widetilde{V}$ in configuration c gives the new configuration,

$$\begin{aligned}\tilde{c}_w &= c - \text{outdeg}(c)\, v + \sum_{w\in\widetilde{V}} \text{wt}(v, w)\, w \\ &= c - \widetilde{\Delta}\, v.\end{aligned}$$

When v fires, we imagine $\text{wt}(e)$ grains of sand traveling along each edge e emanating from v and being deposited at e^+. If $e^+ = s$, then sand sent along e disappears down the sink. If c is unstable at v, we say that firing v is *legal*. The sequence of nonsink vertices $u_1, \ldots, u_k$ is a *legal firing sequence* for a configuration c if it is legal to fire u_1 and then it is legal to fire each subsequent u_i from the configuration obtained by firing $u_1, \ldots, u_{i-1}$. The configuration resulting from applying a legal firing sequence to c is the configuration $\tilde{c} = c - \widetilde{\Delta}\, \sigma$ where $\sigma \in \mathbb{Z}\widetilde{V}$ is such that σ_v is the number of times vertex v appears in the sequence. We write

$$c \xrightarrow{\sigma} c - \widetilde{\Delta}\, \sigma.$$

In general, we write $c \to \tilde{c}$ if $\tilde{c}$ is the result of applying a legal firing sequence to c. In this case, since the reduced Laplacian is invertible (by the Matrix-Tree theorem, for instance), there exists a unique $\sigma \in \widetilde{V}$ such that $\tilde{c} = c - \widetilde{\Delta}\, \sigma$. This σ is called the *firing script* or *firing vector* for $c \to \tilde{c}$.

We have the following existence and uniqueness theorem.

Theorem 2.7. *Let c be a sandpile configuration.*

1. *There exists a stable configuration $\tilde{c}$ such that $c \to \tilde{c}$.*
2. *Suppose (i) $c \to \tilde{c}$ and (ii) $c \to \tilde{c}'$. If $\tilde{c}$ is stable, then the number of times each vertex fires in (i) is greater than or equal to the number of times it fires in (ii). If $\tilde{c}'$ is also stable, then $\tilde{c} = \tilde{c}'$.*

Definition 2.8. Let c be a configuration on G. The *stabilization* of a configuration c is the unique stable configuration $\tilde{c}$ such that $c \to \tilde{c}$.

Notation 2.9. The stabilization of a configuration c is denoted c°.

Let $\mathcal{M}$ denote the set of nonnegative stable configurations on G. Then $\mathcal{M}$ is a commutative monoid under *stable addition*

$$a \circledast b := (a + b)^\circ.$$

Thus, the stable addition is vector addition in $\mathbb{N}\widetilde{V}$ followed by stabilization. The identity is the zero configuration.

Definition 2.10. A configuration c is *accessible* if for each configuration a, there exists a configuration b such that $a + b \to c$. A configuration c is *recurrent* if it is nonnegative, accessible, and stable.

Definition 2.11. The *maximal stable configuration* on G is the configuration

$$c_{\max} = \sum_{v \in \widetilde{V}} (\operatorname{outdeg}(v) - 1)v.$$

Proposition 2.12. *A configuration c is recurrent if and only if there exists a configuration $a \geq 0$ such that*

$$c = (a + c_{\max})^{\circ}.$$

It is not hard to see that the recurrent elements form a submonoid of $\mathcal{M}$. In fact, they form a group called the *sandpile group*, denoted $\mathcal{S}(G)$.

Theorem 2.13. *The collection of recurrent configurations of G form a group under stable addition.*

By Proposition 2.12, the sandpile group can be found by a systematically adding sand to $c_{\max}$ and stabilizing. Considering a graph consisting of otherwise unconnected vertices connected into a common sink by edges of various weights, one sees that every finite Abelian group is the sandpile group for some graph.

Example 2.14. The elements of the sandpile group for the sandpile graph in Figure 1 are listed below using the notation $(c_1, c_2, c_3) := c_1v_1 + c_2v_2 + c_3v_3$:

$$\begin{array}{ccccccc}
(3,3,4) & (3,3,3) & (3,2,4) & (2,3,4) & (3,3,2) & (3,2,3) & (2,3,3) \\
(3,1,4) & (2,2,4) & (1,3,4) & (3,2,2) & (2,2,3) & (1,3,3) & (3,0,4) \\
(2,1,4) & (1,2,4) & (0,3,4) & (1,2,3) & (0,3,3) & (2,0,4) & (1,1,4)
\end{array}$$

Although the zero configuration is the identity for $\mathcal{M}$, it is seldom the identity for $\mathcal{S}(G)$. The following is an easy exercise.

Proposition 2.15. *The following are equivalent:*

1. *the zero-configuration $\vec{0}$ is recurrent;*
2. *every stable configuration is recurrent;*
3. *Every cycle of G passes through the sink vertex.*

We now give another description of the sandpile group.

Definition 2.16. The *Laplacian lattice*, $\mathcal{L} \subset \mathbb{Z}V$, is the image of Δ. The reduced Laplacian lattice, $\widetilde{\mathcal{L}} \subset \mathbb{Z}\widetilde{V}$, is the image of $\widetilde{\Delta}$. The *critical group* for G is

$$\mathcal{C}(G) = \mathbb{Z}\widetilde{V}/\widetilde{\mathcal{L}}.$$

Theorem 2.17. *There is an isomorphism of Abelian groups*

$$\begin{array}{ccc}
\mathcal{S}(G) & \to & \mathcal{C}(G) \\
c & \mapsto & c + \widetilde{\mathcal{L}}
\end{array}$$

Thus, each element of $\mathbb{Z}\widetilde{V}$ is equivalent to a unique recurrent element modulo the reduced Laplacian lattice. The identity of the sandpile group is the recurrent configuration in $\widetilde{\mathcal{L}}$. It can be calculated as

$$\varepsilon = ((c_{\max} - (2c_{\max})^{\circ}) + c_{\max})^{\circ}.$$

Note that $\varepsilon = 0 \bmod \widetilde{\mathcal{L}}$, and since $c_{\max} - (2c_{\max})^{\circ} \geq 0$, Proposition 2.12 guarantees that ε is recurrent.

Example 2.18. The reduced Laplacian of the sandpile graph in Figure 1 is

$$\widetilde{\Delta} = \begin{pmatrix} 4 & -2 & -3 \\ -1 & 4 & -1 \\ -3 & -1 & 5 \end{pmatrix}.$$

The Smith normal form of $\widetilde{\Delta}$ is $\mathrm{diag}(1,1,21)$. Hence, $\mathcal{S}(G) \approx \mathbb{Z}/12\mathbb{Z}$. The identity is $(3,1,4)$, computed as follows:

$$\begin{aligned}(c_{\max} - (2c_{\max})^\circ) + c_{\max} &= ((3,3,4) - (6,6,8)^\circ) + (3,3,4)\\ &= ((3,3,4) - (2,0,4)) + (3,3,4)\\ &= (4,6,4) \rightsquigarrow (3,1,4).\end{aligned}$$

As a consequence of the Matrix-Tree theorem, we have the following.

Corollary 2.19. *The order of $\mathcal{S}(G)$ is the sum of the weights of G's directed spanning trees into s.*

Remark 2.20. Babai [1] has noted another characterization of the sandpile group: it is the principal semi-ideal in $\mathcal{M}$ generated by $c_{\max}$, which turns out to be the intersection of all the semi-ideals of $\mathcal{M}$.

Remark 2.21. In the literature, a sandpile configuration is often taken to be an element of $\mathbb{Z}^{\widetilde{V}}$. We prefer to work in the dual group $\mathbb{Z}\widetilde{V} = \mathrm{Hom}(\mathbb{Z}^{\widetilde{V}}, \mathbb{Z})$ for categorical reasons. Suppose that $G = (V, E, s)$ and $G' = (V', E', s')$ are sandpile graphs with reduced Laplacian lattices $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}'$, respectively. Let $\Psi : G' \to G$ be a mapping of graphs that maps s' to s. Applying $\hom_{\mathbb{Z}}(\,\cdot\,, \mathbb{Z})$ to the natural induced map $\mathbb{Z}^G \to \mathbb{Z}^{G'}$ yields $\Psi_* \colon \mathbb{Z}V' \to \mathbb{Z}V$. If $\Psi(\widetilde{\mathcal{L}}') \subseteq \mathcal{L}$, there is an induced mapping of sandpile groups. This would seem to be a reasonable set of morphisms, then, for a category of sandpile groups. For work on the category theory of sandpile groups see [3] and [22].

2.3. **Superstables.** Let $c = u + v$ be a configuration on the (unweighted, undirected) sandpile graph in Figure 2 with sink s. The vertices u and v are both

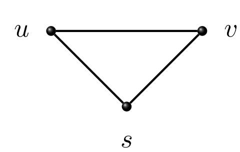


FIGURE 2. Graph G.

stable in c, so there are no legal vertex firings: firing either vertex would result in a negative amount of sand on a vertex. However, firing both vertices simultaneously results in a nonnegative configuration: the zero configuration. Each nonsink vertex would lose two grains of sand, but each would also gain a grain from the other.

Definition 2.22. Let c be a configuration on the sandpile graph $G = (V, E, s)$. A *script-firing*, also called a *cluster-* or *multiset-firing*, with *(firing) script* $\sigma \in \mathbb{N}\widetilde{V}$ is the operation that replaces c with $c - \widetilde{\Delta}\,\sigma$. The script-firing is *legal* if $\sigma \gneq 0$ and $(c - \widetilde{\Delta}\,\sigma)_v \geq 0$ for each $v \in \mathrm{supp}(\sigma)$. Thus, if $c \geq 0$, the script-firing with script $\sigma \gneq 0$ is legal if and only if $c - \widetilde{\Delta}\,\sigma \geq 0$.

A configuration c is *superstable* if c is nonnegative and has no legal script-firings.

Remark 2.23. A superstable configuration is also known as a *G-parking function* [19].

2.4. **Burning configurations.** Speer's script algorithm [20] generalizes the burning algorithm of Dhar, testing whether a configuration is recurrent. We present a variation on Speer's algorithm using burning configurations. A proof for this result appears in the class notes for the Topics in Algebra class, available online [18].

Definition 2.24. A configuration b is a *burning configuration* if it has the following three properties:

(1) $b \in \widetilde{\mathcal{L}}$
(2) $b \geq 0$, i.e., b is a configuration;
(3) for all $v \in \widetilde{V}$, there exists a path to v from some element of $\mathrm{supp}(b)$.

If b is a burning configuration, we call $\sigma_b = (\widetilde{\Delta})^{-1}b$ the *script* or the *firing vector* for b.

Theorem 2.25. *Let b be the burning configuration with script σ_b. Then*

(1) *$(kb)^\circ$ is the identity configuration for $k \gg 0$.*
(2) *A configuration c is recurrent if and only if the stabilization of $c+b$ is c.*
(3) *A configuration c is recurrent if and only if the firing vector for the stabilization of $b+c$ is σ_b.*
(4) *$\sigma_b \geq \vec{1}$.*
(5) *If c is a configuration and τ is the firing vector for the stabilization of $c+b$, then $\tau \leq \sigma_b$.*

Proof. (*** The proof of this will appear in my book on sandpiles. Do I need to include it here? Probably a online reference to a draft of the book is enough ***) □

Theorem 2.26. *There exists a unique burning configuration b with script $\sigma_b = \widetilde{\Delta}^{-1}b$ having the following property: if $\sigma_{b'}$ is the script for a burning configuration b', then $\sigma_{b'} \geq \sigma_b$. For this b, we have:*

(1) *For all $v \in \widetilde{V}$, $b_v < \mathrm{outdeg}(v)$ unless v is a source, i.e., $\mathrm{indeg}(v) = 0$, in which case $b_v = \mathrm{outdeg}(v)$. Thus, b is stable unless G has a source, and in any case, $b_v \leq \mathrm{outdeg}(v)$ for all v.*
(2) *$\sigma_b \geq \vec{1}$ with equality if and only if G has no "selfish" vertices, i.e., no vertex $v \in \widetilde{V}$ with $\mathrm{indeg}(v) > \mathrm{outdeg}(v)$.*

We call this b the minimal burning configuration *and its script, σ_b, the* minimal burning script.

Remark 2.27. To compute the minimal burning configuration, start with b equal to the sum of the columns of $\widetilde{\Delta}$. If $b \geq 0$, stop. Otherwise, if $b_v < 0$ for some $v \in \widetilde{V}$, replace b by $b + \widetilde{\Delta}(v)$. Repeat until $b \geq 0$.

Example 2.28. We would like to compute the minimal burning configuration and corresponding script for the sandpile graph G in Figure 1. Continuing Example 2.18, the sum of the columns of $\widetilde{\Delta}$ is $(-1, 2, 1)^t$. Since the first entry of the sum is negative, add in the first column of $\widetilde{\Delta}$ to get $(3, 1, -2)^t$. Since the third entry is now negative, add in the third column of $\widetilde{\Delta}$ to get $(0, 0, 3)$. Thus, the minimal burning configuration is $b = (0, 0, 3)$, and the burning script is $\sigma_b = (2, 1, 2)$ recording the columns of $\widetilde{\Delta}$ used to obtain b.

3. Lattice ideals

Our reference for this section is [15]. Let A be a finitely generated Abelian group, and let $a_1,\dots,a_n$ be a collection of elements generating A. Define $\phi\colon \mathbb{Z}^n \to A$ by $\phi(e_i) = a_i$, and denote its kernel by Λ. Let $\{t_a : a \in A\}$ be indeterminates, and let

$$\mathbb{C}[A] = \operatorname{Span}_{\mathbb{C}}\{t_a : a \in A\}$$

be the group algebra of A; hence, $t_a t_b = t_{a+b}$ for elements $a, b \in A$. Let $R := \mathbb{C}[x_1,\dots,x_n]$, and define a surjection of rings

$$\begin{array}{rcl} \psi\colon R & \to & \mathbb{C}[A] \\ x_i & \mapsto & t_{a_i}. \end{array}$$

For $c \in \mathbb{N}^n$, we define $x^c = \prod_i x_i^{c_i}$. Then, $\psi(x^c)$ is the group algebra element t_b where $b = \sum_{i=1}^n a_{c_i}$.

For $u \in \mathbb{Z}^n$, we write $u = u^+ - u^-$ with $u^+, u^- \in \mathbb{N}^n$ having disjoint support.

Theorem 3.1.

(1) *The kernel of ψ is the* lattice ideal

$$I(\Lambda) := \operatorname{Span}_{\mathbb{C}}\{x^u - x^v : u, v \in \mathbb{N}^n, u - v \in \Lambda\}.$$

(The vector space span, above, forms an ideal.) Hence, ψ induces an isomorphism $R/I(\Lambda) \approx \mathbb{C}[A]$.

(2) *If $\ell_1,\dots,\ell_k$ are generators for the $\mathbb{Z}$-module, Λ, then $I(\Lambda)$ is the saturation of*

$$J = \langle x^{\ell_i^+} - x^{\ell_i^-} : i = 1,\dots,k\rangle$$

with respect to the ideal generated by the product of the indeterminates, $\prod_{i=1}^n x_i$. Thus,

$$I(\Lambda) = \{f \in R : (\textstyle\prod_{i=1}^n x_i)^m f \in J \textit{ for some } m \in \mathbb{N}\}.$$

(3) *The Krull dimension of $R/I(\Lambda)$ is the rank of the $\mathbb{Z}$-module Λ.*

Let $U \subset \mathbb{N}^n$ such that $X := \{x^u : u \in U\}$ is a $\mathbb{C}$-vector space basis for $R/I(\Lambda)$. Letting $g := (a_1,\dots,a_n) \in A^n$,

$$\psi(X) = \{t_{u\cdot g} : u \in U\} = \{t_a : a \in A\},$$

the last equality holding since $R/I(\Lambda)$ and $\mathbb{C}[A]$ are isomorphic as vector spaces via ψ. Thus, ψ induces a bijection of X with A, which endows X with the structure of a group isomorphic to A. For $u, v \in U$, we define $x^u x^v = x^w$ where w is the unique element of U for which $w \cdot g = (u + v) \cdot g$.

A choice of a monomial ordering on R gives a natural choice for U, namely, those $u \in \mathbb{N}^n$ such that x^u is not divisible by the initial term of any element of $I(\Lambda)$, e.g., not divisible by the initial term of any element of a Gröbner basis for $I(\Lambda)$. This will be discussed in §5.

4. Toppling ideals

Let G be a sandpile graph, and identify the vertices with $\{1,\dots,n+1\}$ with $n+1$ representing the sink. To avoid ambiguity, we will sometimes denote vertex i by v_i. By ordering the vertices, we thus have the exact sequence for the sandpile group of G,

$$0 \to \mathbb{Z}^n \xrightarrow{\tilde{\Delta}} \mathbb{Z}^n \to \mathcal{S}(G) \to 0.$$

Recall our notation for the reduced Laplacian lattice:

$$\widetilde{\mathcal{L}} = \operatorname{im}(\widetilde{\Delta}) = \ker(\mathbb{Z}^n \to \mathcal{S}(G)).$$

Definition 4.1. The *toppling ideal* for G is the lattice ideal for $\widetilde{\mathcal{L}}$,

$$I(G) := \operatorname{Span}_{\mathbb{C}}\{x^u - x^v : u = v \bmod \widetilde{\mathcal{L}}\} \subset R = \mathbb{C}[x_1, \ldots, x_n].$$

The *coordinate ring* for G is $R/I(G)$.

Thus,

$$R/I(G) \approx \mathbb{C}[\mathcal{S}(G)].$$

For each nonsink vertex i, define the *toppling polynomial*

$$t_i = x_i^{d_i - \mathrm{wt}(i,i)} - \textstyle\prod_{j \neq i} x_j^{\mathrm{wt}(i,j)}.$$

Proposition 4.2. *The ideal $I(G)$ is generated by the toppling polynomials, $\{t_i\}_{i=1}^n$, and the polynomial $x^b - 1$ where b is any burning configuration.*

Proof. Let $J = (t_i : i = 1, \ldots, n) + (x^b - 1)$. It is clear that $J \subseteq I(G)$, and by Theorem 3.1, Part (2), $I(G)$ is the saturation of J with respect to the ideal $(x_1 \cdots x_n)$. So it suffices to show that J is already saturated with respect to that ideal. Suppose that $(x_1 \cdots x_n)^k f \in J$ for some $f \in R$ and for some k. For each positive integer m consider the monomial x^{mb}. We think of this monomial as a configuration of sand with mb_i grains of sand on vertex i. If vertex i of this configuration is unstable, we think of firing the vertex as replacing $x_i^{mb_i}$ by $x_i^{mb_i - d_i} \prod_{j \neq i} x_j^{\mathrm{wt}(i,j)}$. Performing this replacement in x^{mb} gives an equivalent monomial modulo J. Recall that every vertex of G is connected by a directed path from a vertex in the support of b. Thus, by taking m large enough and firing appropriate vertices, we arrive at a monomial x^γ, equivalent to x^{mb} modulo J and corresponding to a configuration with at least k grains of sand at each vertex. Write $x^\gamma = x^\delta (x_1 \cdots x_n)^k$ for some monomial x^δ. Modulo J, we have

$$\begin{aligned} 0 &= (x_1 \cdots x_n)^k f \\ &= x^\delta (x_1 \cdots x_n)^k f \\ &= x^\gamma f \\ &= x^{mb} f \\ &= f. \end{aligned}$$

Thus, $f \in J$, as required. □

Remark 4.3. As in the proof of the above theorem, we can identify a monomial x^a with the configuration a on G. If $a \to b$ as sandpile configurations, then $x^a = x^b$ in $R/I(G)$.

Remark 4.4. The toppling ideal was introduced by Cori, Rossin, and Salvy [5]. They considered only undirected graphs and defined the ideal via generators. For an undirected graph, the all-1s vector is a burning script; so Proposition 4.2 shows that our definition coincides with theirs in the case of an undirected graph.

Example 4.5. The sandpile graph G in Figure 3 has an undirected edge connecting v_1 and v_2. All edges have weight 1 except (v_3, v_2), which has weight 2. The

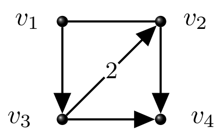


FIGURE 3. Sandpile graph G with sink v_4.

graph G has a burning script $\sigma = (1, 2, 1)$ and corresponding burning configuration $b = (0, 1, 2)$. Thus,

$$I(G) = (x_1^2 - x_2x_3, x_2^2 - x_1, x_3^3 - x_2^2, x_2x_3^2 - 1).$$

Definition 4.6. Let $f \in R = \mathbb{C}[x_1, \dots, n]$, and let x_{n+1} be another indeterminate. The *homogenization* of f with respect to x_{n+1} is the homogeneous polynomial

$$f^h := x_{n+1}^{\deg f} f\left(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}\right).$$

If $I \subseteq R$ is an ideal, the *homogenization* of I with respect to x_{n+1} is the ideal

$$I^h := (f^h : f \in I).$$

Now consider the exact sequence corresponding to the full Laplacian,

$$\mathbb{Z}^{n+1} \xrightarrow{\Delta} \mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1}/\mathcal{L} \to 0$$

recalling the notation for the Laplacian lattice, $\mathcal{L} := \operatorname{im}(\Delta)$. Let $S = \mathbb{C}[x_1, \dots, x_{n+1}]$ and consider the lattice ideal for $\mathcal{L}$.

Definition 4.7. The *homogeneous toppling ideal* for G is

$$I_h(G) := \operatorname{Span}_{\mathbb{C}}\{x^u - x^v : u = v \bmod \mathcal{L}\} \subset S = \mathbb{C}[x_1, \dots, x_{n+1}].$$

The *homogeneous coordinate ring* for G is $S/I_h(G)$.

The following proposition is straightforward. Its hypothesis is satisfied for any Eulerian graph, i.e., a graph for which $\operatorname{indeg}(v) = \operatorname{outdeg}(v)$ for each vertex v. For example, it holds for any undirected graph. Moreover, given any sandpile graph with sink s, removing all out-edges from s creates a new sandpile graph with the same sandpile group and for which the hypothesis of the proposition hold.

Proposition 4.8. *If* $\Delta(v_{n+1}) \in \operatorname{Span}\{\Delta(v_1), \dots, \Delta(v_n)\}$, *then* $I_h(G) = I(G)^h$.

Example 4.9. The graph G in Figure 4 does not satisfy the hypotheses of Proposition 4.8. Regarded as a sandpile graph with sink v_1, the toppling ideal for G is $(x_1^2 - 1)$. As a sandpile graph with sink v_2, its toppling ideal is $(x_2^3 - 1)$. Its homogeneous toppling ideal is $I_h(G) = (x_1 - x_2)$, equivalent to that of the undirected graph with a single edge connecting v_1 and v_2 (or to that of the directed graph consisting of a single directed edge connecting v_1 to v_2).

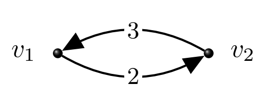


FIGURE 4. Graph G.

Remark 4.10. In general, homogenizing the generators of an ideal does not produce a complete set of generators for the homogenized ideal. For instance, in Example (***FIX: Refer to the example in the section on resolutions***) the toppling ideal is generated by 4 polynomials, whereas its homogeneous toppling ideal is minimally generated by 6 polynomials.

Theorem 4.11. *Let $\widetilde{\mathcal{L}}$ be any submodule of $\mathbb{Z}^n$ having rank n. Then there exists a sandpile graph whose reduced Laplacian lattice is $\widetilde{\mathcal{L}}$. Every lattice ideal defining a finite set of points is the lattice ideal associated with the reduced Laplacian of some sandpile graph.*

Proof. In light of part 3 of Theorem 3.1, it suffices to prove that given an $n \times n$ matrix M of rank n, there exists a matrix M' with the same integer column span as M and which is the reduced Laplacian matrix of some sandpile graph. Recall that a matrix M' is the reduced Laplacian of a directed multigraph if and only if (i) $\deg(c) \geq 0$ for each column c of M', (ii) $M'_{ii} > 0$, (iii) $M'_{ij} \leq 0$ for $i \neq j$. (If c is a column vector of a matrix, then $\deg(c)$ is the sum of the entries of c.) If in addition M' has full rank, then it is a sandpile graph, i.e., it has a globally accessible sink, by the Matrix-Tree Theorem. The desired matrix M' is produced by Algorithm 4.12, stated below. It proceeds in three steps, modifying the columns of M using only invertible integral column operations.

First, since M has rank n, not all columns have $\deg(c) = 0$. Using the Euclidean algorithm, by adding multiples of one column to another, we set $\deg(c)$ to 0 for all but one column c of M (line 1). By possibly moving and negating that column, we have that $\deg(c_i) = 0$ for all but the first column c_1, for which $\deg(c_1) > 0$.

Next, we repeat the Euclidean algorithm another $(n-2)$ times, now on the superdiagonal entries of each of the first $(n-2)$ rows in turn (lines 2–9). Again by adding multiples of one column to another, we have every entry more than one row above the diagonal set to 0. Note that since this step only involves addition of columns whose degree is already zero, the column degrees are not affected. Additionally, since M had rank n and the last $(n-1)$ columns have degree zero, we have that each of these columns has a nonzero superdiagonal entry. Now by negating columns where necessary, we may assume that the nonzero superdiagonal entry of each column is negative.

At this point, the last column satisfies (i)–(iii). Assuming the last r columns $c_{n-r+1}, \ldots, c_n$ satisfy (i)–(iii) for $r \leq n-2$, we claim that for any $1 \leq t \leq r$ there is a vector $v^t \in \operatorname{span}_{\mathbb{Z}}\{c_{n-r+1}, \ldots, c_n\}$ with $v^t_n < 0$, $v^t_{n-t} > 0$, and all other entries zero. For $r = 1$, the vector v^1 is obtained by negating c_n, so we proceed by induction on r. With the hypotheses satisfied for some r, we already have appropriate vectors $v^1, \ldots, v^{r-1}$. To obtain v^r, note that $-c_{n-r+1}$ has a positive entry in row $(n-r)$, so by adding appropriate multiples of the v^t for $t < r$ we obtain the desired column vector.

Given that such vectors v^t exist, it is clear that we may iteratively correct the columns from right to left by adding multiples of the higher indexed columns. We now give this algorithm explicitly. In what follows, $v[j]$ denotes the j-th entry of the column vector v, and the Euclidean algorithm terminates when run in-place on some set of integers, S, once a single element of S equals the positive GCD of the elements of S and every other element of S is zero.

Algorithm 4.12.

Input: An $n \times n$ matrix M of rank n with columns $c_1, \dots c_n$.
Output: The reduced Laplacian matrix $\widetilde{\Delta}(G)$ of a directed multigraph G such that $\widetilde{\Delta}(G) = MU$ for some invertible integral matrix U.

1 Run the Euclidean algorithm on the set $S = \{\deg(c_k)\}$ by subtracting one column from another at each step. Swap columns so that $\deg(c_1) = \gcd(S)$ and $\deg(c_i) = 0$ for $i > 1$.
2 **for** $k \leftarrow 2$ **up to** $n-1$ **do**
3   Run the Euclidean algorithm on the set $S = \{c_i[k-1] : i \geq k\}$ by subtracting one column from another at each step.
4   Swap columns so that $c_k[k-1] = \gcd(S)$ and $c_i[k-1] = 0$ for $i > k$
5    $c_k \leftarrow -c_k$
6 **end for**
7 **if** $c_n[n-1] > 0$ **then**
8    $c_n \leftarrow -c_n$
9 **end if**
10 **for** $k \leftarrow n-1$ **down to** 1 **do**
11   **for** $i \leftarrow k+2$ **up to** n **do** // this loop is not entered until $k \leq n-2$
12   **while** $c_k[i-1] > 0$ **do**
13          $c_k \leftarrow c_k + c_i$
14   **end while**
15   **end for**
16    $v \leftarrow -c_{k+1}$
17   **for** $i \leftarrow k+2$ **up to** n **do** // this loop is not entered until $k \leq n-2$
18       $v \leftarrow |c_i[i-1]| \cdot v + v[i-1] \cdot c_i$
19   **end for**
20   **while** $c_k[k] \leq 0$ or $c_k[n] > 0$ **do**
21       $c_k \leftarrow c_k + v$
22   **end while**
23 **end for**
24 **return** $[c_1 \cdots c_n]$

□

For the sake of the following corollary, a *weighted path graph*, $P = u_1 \dots u_k$ is a graph with vertex set $\{u_1, \dots, u_k\}$ and weighted edges $\{(u_i, u_{i+1}) : 1 \leq i < k\}$. If F and F' are weighted digraphs, their *graph sum* is the graph $F + F'$ whose weighted adjacency matrix is the sum of those for F and F'.

Corollary 4.13. *Let G be a sandpile graph with vertex set $V = \{v_1, \dots, v_{n+1}\}$ and sink v_{n+1}. Then there exists a weighted path graph $P = v_n v_{n-1} \cdots v_1 v_{n+1}$ and a directed acyclic graph D on the nonsking vertices $\widetilde{V}$ oriented from lower-indexed vertices to higher such that the graph sum $G' = P + D$ has the same Laplacian lattice as G.*

The above simply states the form of the graph given by the output of Algorithm 4.12. The graph G' of Corollary 4.13 is not uniquely determined. For instance, by iterating line 21 of Algorithm 4.12 more times than necessary, one may generate inifinitely many graphs G' of the form described in the corollary, each with Laplacian lattice $\mathcal{L}$.

Example 4.14. One sandpile graph of the form given by Corollary 4.13 with the same Laplacian lattice as the sandpile graph G from example 4.5 is G' appearing in Figure 5.

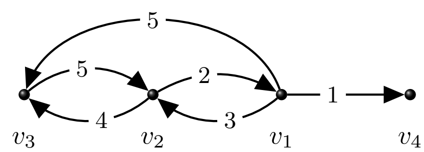


FIGURE 5. The sandpile graph G' for Example 4.14.

Question 4.15. When is it the case that a submodule of $\mathbb{Z}^n$ with rank n is the reduced Laplacian lattice of an *undirected* graph? It is not always the case. For instance, Figure 7 is a directed sandpile graph whose lattice ideal is Gorenstein (cf. §8) and with sandpile group of order 5. By Theorem (***fill this in***), any undirected graph with Gorenstein lattice ideal must be a tree and would thus have sandpile group of order 1.

5. Gröbner bases of toppling ideals

We recommend [6] as a general reference for the theory of Gröbner bases needed in this section. Let $R = \mathbb{C}[x_1, \dots, x_n]$.

Definition 5.1. A *monomial order*, $>$, on R is a total ordering on the monomials of R satisfying

(1) If $x^a > x^b$, then $x^{c+a} > x^{c+b}$ for all $c \geq 0$;
(2) $1 = x^0$ is the smallest monomial.

Example 5.2. The following are the most common examples of monomial orders:

(1) Lexicographic ordering, `lex`, is defined by $x^a > x^b$ if the left-most nonzero entry of $a - b$ is positive (i.e., more of the earlier indeterminates).
(2) Degree lexicographic ordering, `deglex`, is defined by $x^a > x > b$ if $\deg(a) > \deg(b)$ or if $\deg(a) = \deg(b)$ and the left-most nonzero entry of $a - b$ is positive (i.e., order by degree and break ties with `lex`).
(3) Degree reverse lexicographic ordering, `grevlex`, is defined by $x^a > x^b$ if $\deg(a) > \deg(b)$ or if $\deg(a) = \deg(b)$ and the right-most nonzero entry of $a - b$ is negative (i.e., order by degree then break ties by checking which monomial has fewer of the later indeterminates).

A monomial multiplied by a constant is called a *term*. Once a monomial ordering is fixed, write $\alpha\, x^a > \beta\, x^b$ for two terms if α and β are nonzero and $x^a > x^b$. Each $f \in R$ is a sum of terms corresponding to distinct monomials. We denote the leading term—the largest term with respect to the chosen monomial ordering—by $\mathrm{LT}(f)$.

Definition 5.3. Fix a monomial ordering on R and let $f, g \in R$. The *S-polynomial* for the pair (f, g) is

$$S(f,g) = \frac{\mathrm{lcm}(\mathrm{LT}(f), \mathrm{LT}(g))}{\mathrm{LT}(f)}\, f - \frac{\mathrm{lcm}(\mathrm{LT}(f), \mathrm{LT}(g))}{\mathrm{LT}(g)}\, g.$$

Definition 5.4. Fix a monomial ordering on R, and let I be an ideal of R. A finite subset Γ of I is a *Gröbner basis* for I with respect to the given monomial ordering if any of the following equivalent conditions hold:

(1) $(\mathrm{LT}(g) : g \in \Gamma) = (\mathrm{LT}(f) : f \in I)$.
(2) For all $f \in I$, there is a $g \in \Gamma$ such that $\mathrm{LT}(g)$ divides $\mathrm{LT}(f)$.
(3) Each $f \in I$ may be reduced to 0 by Γ, i.e., by repeatedly reducing by elements of Γ.
(4) For all $g, g' \in \Gamma$, the S-polynomial $S(g, g')$ reduces to 0 by Γ and Γ is a generating set for I.

The last criterion is essentially Buchberger's algorithm for calculating a Gröbner basis.

Let $\Gamma = \{g_1, \ldots, g_m\}$ be the Gröbner basis for an ideal $I \subseteq R$ with respect to some monomial ordering, and let $f \in \mathbb{R}$. If f has a term m divisible by $\mathrm{LT}(g_i)$ for some i, then replace f by $f - \frac{m}{\mathrm{LT}(g_i)}\, g_i$. Repeating this process one arrives at a remainder r that is unique with respect to the property that (i) $r = f + g$ for some $g \in I$ and (ii) r has no terms divisible by any leading term of an element of Γ. We call this remainder the *reduction* or *normal form* of f with respect to the Gröbner basis Γ.

Notation 5.5. The reduction of f with respect to Γ is denoted by $f \,\%\, \Gamma$. If $g \in R$, we write $f \,\%\, g$ for the special case in which $I = (g)$ and $\Gamma = \{g\}$.

Definition 5.6. Fix a monomial ordering on R and let I be an ideal of R. The set of monomials of R that are not divisible by the leading term of a Gröbner basis element for I with respect to the given ordering is called the *normal basis* for R/I.

By Macaulay's theorem, a normal basis is a vector space basis for R/I.

We now introduce an appropriate monomial ordering for sandpiles, due to Cori, Rossin, and Salvy, [5]. Let G be a sandpile graph as in §4.

Definition 5.7. Let G be a sandpile graph with vertices $\{v_1, \ldots, v_{n+1}\}$ and with sink v_{n+1}. A *sandpile monomial ordering* on $R = \mathbb{C}[x_1, \ldots, x_n]$ is any `grevlex` ordering for which $x_i > x_j$ if the length of the shortest path from vertex v_j to the sink is no greater than that for v_i. Given a sandpile monomial ordering $>$ on R, the sandpile monomial ordering on $S = \mathbb{C}[x_1, \ldots, x_{n+1}]$ *compatible* with $>$ is the `grevlex` order extending $>$ for which $x_i > x_{n+1}$ for $i = 1, \ldots, n$.

Proposition 5.8. *With notation as in Definition 5.7, let $>$ be a sandpile monomial ordering on R, extended to a compatible sandpile monomial ordering on S. Let $I \subset R$ be the toppling for G.*

(1) *Let Γ a Gröbner basis for I with respect to $>$, and let Γ^h be the subset of S formed by homogenizing each element of Γ. Then Γ^h is a Gröbner basis for the homogenization $I^h \subset S$.*
(2) *The normal bases for R/I and for $S/(I^h + (x_{n+1})$ consist of the same set of monomials. Hence, R/i and $S/(I^h + (x_{n+1}))$ are isomorphic as vector spaces.*

Proof. The first part of the proposition is a general result for `grevlex` orderings (cf. Exercise 5, §8.4, [6]). It is straightforward to check that if $f \in R$, then $\mathrm{LT}(f) = \mathrm{LT}(f^h)$, from which the second part follows. □

ASSUMPTION: For the rest of this section, we fix a sandpile monomial ordering on R and assume the vertices are numbered so that $x_i > x_j$ if $i < j$.

The utility of the above convention becomes apparent when one considers topplings of sandpiles.

Proposition 5.9. *Let $a, b \in \mathbb{N}\widetilde{V}$ be distinct configurations on G such that $a \to b$, i.e., b is obtained from a by a sequence of vertex firings. Then, $x^a > x^b$ with respect to the sandpile monomial ordering we have fixed on R.*

Proof. Each vertex firing deceases the size of the corresponding monomial. The reason is that either the vertex firing shoots sand into the sink, decreasing the total degree of the corresponding monomial, or shoots sand to a vertex closer to the sink, in which case the corresponding monomial has more of the later indeterminates. □

We now proceed to compute a Gröbner basis for the toppling ideal. Let

$$\begin{array}{rcl} E\colon \mathbb{Z}\widetilde{V} & \to & R \\ \ell & \mapsto & x^{\ell^+} - x^{\ell^-}. \end{array}$$

Then define $\mathcal{T} = E \circ \widetilde{\Delta} : \mathbb{Z}\widetilde{V} \to R$. Thus, $\mathcal{T}(v_i)$ is the i-th toppling polynomial, defined earlier, and for any configuration c, we have $x^c \,\%\, \mathcal{T}(v_i) = x^{c'}$ where c' is the configuration obtained from c by firing v_i until v_i is stable. Morever, if σ is a firing-script, then $x^c \,\%\, \mathcal{T}(\sigma)$ yields the monomial corresponding to the configuration formed by firing σ as many times as legal from c.

Theorem 5.10. *Let b be a burning configuration, and let σ_b be its script. Then*

$$\Gamma_b = \{\mathcal{T}(\sigma) : 0 \leq \sigma \leq \sigma_b\}$$

is a Gröbner basis for $I(G)$.

Proof. We have $\mathrm{im}(\mathcal{T}) \subset I(G)$ by definition of $I(G)$. On the other hand, $\mathcal{T}(v_i)$ is the i-th toppling polynomial and $\mathcal{T}(\sigma_b) = x^b - 1$. So $I(G) = \mathrm{Span}_{\mathbb{C}}\{\mathrm{im}(\mathcal{T})\}$ by Proposition 4.2.

We need to show that all S-polynomials of Γ_b reduce to 0 by Γ_b. Let σ_1 and σ_2 be scripts with $\sigma_1, \sigma_2 \leq \sigma_b$. Write

$$\mathcal{T}(\sigma_i) = x^{c_i^+} - x^{c_i^-}$$

for $i = 1, 2$ where c_i^- is the configuration obtained from c_i^+ by firing script σ_i. Hence, $x^{c_i^+}$ is the leading term of $\mathcal{T}(\sigma_i)$ for each i. Define

$$x^{a_i} = \frac{\mathrm{lcm}(x^{c_1^+}, x^{c_2^+})}{x^{c_i^+}}$$

for $i = 1, 2$ so that $a_1 + c_1^+ = a_2 + c_2^+ = c$ for some configuration c. We must show that the S-polynomial,

$$\begin{array}{rcl} S(\sigma_1, \sigma_2) & = & x^{a_1}\mathcal{T}(\sigma_1) - x^{a_2}\mathcal{T}(\sigma_2) \\ & = & x^{a_2 + c_2^-} - x^{a_1 + c_1^-}, \end{array}$$

reduces to 0. Since both scripts σ_1 and σ_2 are legal from c, so is the script $\sigma = \max(\sigma_1, \sigma_2)$ defined by $\sigma_v = \max(\sigma_{1,v}, \sigma_{2,v})$. Note that $\sigma \leq \sigma_b$. Letting c' be the configuration obtained by firing σ, we have the sequence of script firings

$$a_i + c_i^+ \xrightarrow{\sigma_i} a_i + c_i^- \xrightarrow{\sigma - \sigma_i} c'$$

for $i = 1, 2$, which shows that the S-polynomial reduces to 0 using the elements $\mathcal{T}(\sigma - \sigma_i)$ for $i = 1, 2$. □

Remark 5.11. In the case of an undirected graph, one may take the burning script to be the vector whose components are all ones. Thus, the script firings that are relevant in constructing the Gröbner basis, described in the statement of the previous theorem, can be identified with firing subsets of vertices (none more than once). The paper [5] goes further, in this case, to describe a *minimal* Gröbner basis, i.e., one in which each member has the property that none if its terms is divisible by the leading term any other member. It consists of the subset of the Gröbner basis elements described in the previous theorem corresponding to $X \subseteq \widetilde{V}$ such that the subgraphs of G induced by X and by $\widetilde{V} \setminus X$ are each connected. It would be interesting to see if this result could be generalized to the case of directed graphs.

Theorem 5.12. *Each nonnegative configuration is equivalent to a unique superstable sandpile modulo $\widetilde{\mathcal{L}}$, and*

$$\{x^c : c \text{ is a superstable configuration}\}$$

is the normal basis for $R/I(G)$ with respect to the sandpile monomial ordering.

Proof. Two nonnegative configurations are equivalent modulo $\widetilde{\mathcal{L}}$ if and only if their corresponding monomials are equivalent modulo the toppling ideal, $I(G)$. In detail, first let $c_1, c_2 \in \mathbb{N}^n$ and suppose

$$c_1 - c_2 = \ell = \ell^+ - \ell^- \in \widetilde{\mathcal{L}}.$$

Then $c_1 \geq \ell^+$ and $c_2 \geq \ell^-$. Define $e = c_1 - \ell^+ = c_2 - \ell^- \geq 0$. Then

$$x^{c_1} - x^{c_2} = x^e(x^{\ell^+} - x^{\ell^-}) \in I(G).$$

Conversely, suppose $x^{c_1} - x^{c_2} \in I(G)$. Identify the sandpile group, $\mathcal{S}(G)$ with $\mathbb{Z}^n/\widetilde{\mathcal{L}}$. Let

$$\begin{aligned} \psi\colon \mathbb{C}[x_1, \ldots, x_n] &\to \mathbb{C}[\mathbb{Z}^n/\widetilde{\mathcal{L}}] \\ x_i &\mapsto t_{e_i} \end{aligned}$$

be the mapping into the group algebra. Here, e_i is the image of the i-th standard basis vector for $\mathbb{Z}^n$. Then $I(G) = \ker\psi$. Hence,

$$0 = \psi(x^{c_1} - x^{c_2}) = t_{c_1} - t_{c_2}.$$

In other words, $c_1 - c_2 \in \widetilde{\mathcal{L}}$.

Now let c be any nonnegative configuration. Since $x^c \,\%\, \mathcal{T}(\sigma) = x^{c'}$ where c' is obtained by firing the script σ as many times as is legal, the normal form for x^c with respect to the sandpile monomial ordering is superstable. Since the normal form is unique, so is this superstable element. □

Remark 5.13. As noted in §4, we have $R/I(G) \approx \mathbb{C}[\mathcal{S}(G)]$. Hence, by the previous theorem, we see that the sandpile group can be thought of as the set of superstables where the sum of superstables c_1 and c_2 is taken to be $\log(x^{c_1}x^{c_2} \,\%\, I(G))$.

The following can be found in [13] for the case of Eulerian graphs, i.e., if $\text{indeg}(v) = \text{outdeg}(v)$ for each vertex v. Here we extend the result to general sandpile graphs (for which the underlying graph is a directed multigraph).

Corollary 5.14. *A configuration c is superstable if and only if $c_{\max} - c$ is recurrent.*

Proof. By Theorems 5.12 and 2.17, the number of superstable configurations is equal to the number of recurrent configuration. Thus, is suffices to show that if c is superstable, then $c_{\max} - c$ is recurrent.

Let b be a burning configuration for G with burning script σ_b. Since c is superstable, there exists $u_1 \in \operatorname{supp}(\sigma_b)$ such that $(c - \widetilde{\Delta}\,\sigma_b)_{u_1} < 0$. Similarly, there exists $u_2 \in \operatorname{supp}(\sigma_b - u_1)$ such that $(c - \widetilde{\Delta}(\sigma_b - u_1))_{u_2} < 0$. Continuing, we find a sequence of nonsink vertices $u_1, \dots, u_k$ such that $\sum_{i=1}^{k} u_i = \sigma_b$ and for $1 \le j \le k$,

$$\left(c - \widetilde{\Delta}(\sigma_b - \textstyle\sum_{i=1}^{j-1} u_i)\right)_{u_j} < 0.$$

It follows that $u_1, \dots, u_k$ is a legal firing sequence for $c_{\max} - c + b$. Hence, $c_{\max} - c$ is recurrent by Theorem 2.25. $\square$

In light of Corollary 5.14, we say that the superstables are *dual* to the recurrents.

6. Zeros of the toppling ideal

Given any ideal $I \in R = \mathbb{C}[x_1, \dots, x_n]$, the set of zeros of I is

$$Z(I) = \{p \in \mathbb{C}^n : f(p) = 0 \text{ for all } f \in I\}.$$

In this section, our goal is to describe the set of zeros of the toppling ideal.

Proposition 6.1. *Let G be a sandpile graph. Then the set of zeros of its toppling ideal, $I(G)$, is finite.*

Proof. Since $I(G)$ is the lattice ideal for a square matrix of full rank, part 3 guarantees that the set of zeros is finite. However, we will give a direct proof. We have seen that

$$R/I(G) \approx \mathbb{C}[\mathcal{S}(G)],$$

and thus, $R/I(G)$ is a finite-dimensional vector space over C. For each indeterminate $x_i \in R$, consider the powers $1, x_i, x_i^2, \dots$. By finite-dimensionality, the image of these powers in the quotient ring are linearly dependent. This means there is a polynomial f_i such that $f_i(x_i) \in I(G)$. Each f_i will have a finite number of zeros, and thus, for each i, we see that the there are a finite number of possible i-th coordinates for any zero of the toppling ideal. $\square$

Remark 6.2. In fact, the i-th coordinates of the zeros of the toppling ideal are the eigenvalues of the multiplication mapping

$$\begin{array}{ccc} R/I(G) & \to & R/I(G) \\ g & \mapsto & x_i g \end{array}$$

See [7], for instance.

6.1. **Orbits of representations of Abelian groups.**

6.1.1. *Affine case.* Let $\{a_1, \dots, a_n\}$ be generators (not necessarily distinct) for a finite Abelian group, A. Consider the exact sequence

$$\begin{array}{rcl} 0 \to \Lambda \to \mathbb{Z}^n & \to & A \to 0 \\ e_i & \mapsto & a_i \end{array} \tag{6.1}$$

where Λ is defined as the kernel of the given mapping $\mathbb{Z}^n \to A$. Take duals, i.e., applying $\operatorname{Hom}_{\mathbb{Z}}(\,\cdot\,, \mathbb{C}^\times)$, gives the sequence

$$1 \leftarrow \Lambda^* \leftarrow (\mathbb{C}^\times)^n \leftarrow A^* \leftarrow 1, \tag{6.2}$$

where A^* is the character group of A.
Remark 6.3.

(1) Exactness of (6.2) is not immediate. The exactness at $\Lambda^* \leftarrow (\mathbb{C}^\times)^n$ follows because $\mathbb{C}^\times$ is a *divisible* Abelian group. An Abelian group B is divisible if for all $a \in B$ and positive integers n there exists $b \in B$ such that $nb = a$. (So for the multiplicative group $\mathbb{C}^\times$, each element has an n-th root.) Applying $\operatorname{Hom}_{\mathbb{Z}}(\,\cdot\,, B)$ to an exact sequence of Abelian groups ($\mathbb{Z}$-modules) always gives an exact sequence precisely when B is divisible. The proof of this, in general, is not immediate. However, in the case in which we are most concerned, the exactness is easy to establish. Suppose $A = \mathcal{S}(G)$ is the sandpile group of a sandpile graph, and suppose Λ is the reduced Laplacian lattice, $\widetilde{\mathcal{L}} = \operatorname{im}(\widetilde{\Delta}) \hookrightarrow \mathbb{Z}^n$. We would like to show that the natural map, given by composition,
$$\operatorname{Hom}(\mathbb{Z}^n, \mathbb{C}^\times) \to \operatorname{Hom}(\widetilde{\mathcal{L}}, \mathbb{C}^\times)$$
is surjective. Let $\phi\colon \widetilde{\mathcal{L}} \to \mathbb{C}^\times$ be given. Since the reduced Laplacian has full rank, given $v \in \mathbb{Z}^n$, there exist unique rational numbers α_ℓ such that $v = \sum_\ell \alpha_\ell \ell$, with the sum going over a basis for $\widetilde{\mathcal{L}}$ (say, over the columns of the reduced Laplacian). Then define $\tilde{\phi}\colon \mathbb{Z}^n \to \mathbb{C}^\times$ by $\tilde{\phi}(v) = \sum_\ell \phi(\ell)^{\alpha_\ell}$.

(2) To be explicit, denote the mapping $\mathbb{Z}^n \to A$ by ϕ. Then part of sequence (6.2) is
$$\begin{array}{ccccc} A^* & \to & \operatorname{Hom}(\mathbb{Z}^n, \mathbb{C}^\times) & \approx & (\mathbb{C}^\times)^n \\ \chi & \mapsto & \chi \circ \phi & \mapsto & (\chi(a_1), \dots, \chi(a_n)). \end{array}$$

We get an n-dimensional representation of A^*:
$$\rho\colon A^* \to (\mathbb{C}^\times)^N \to \operatorname{GL}(\mathbb{C}^n)$$
given by
$$\rho(\chi) = \operatorname{diag}(\chi(a_1), \dots, \chi(a_n)).$$
In other words, the choice of generators for A induces a homomorphism of A^* into the group of invertible $n \times n$ matrices over $\mathbb{C}$. (Every n-dimensional representation of A^* over $\mathbb{C}$ is a direct sum of characters of A^*, i.e., of elements of $A^{**} \approx A$. So this section can be regarded as saying something about representations of A^*, in general.)

For each $z \in \mathbb{C}^n$, define the *orbit of z under ρ* to be
$$\mathcal{O}_\rho(z) = \{\rho(\chi)z : \chi \in A^*\} = \{(\chi(a_1)z_1, \dots, \chi(a_n)z_n) : \chi \in A^*\}.$$

We will assume that no coordinate of z is zero, in which case by scaling coordinates of $\mathbb{C}^n$, we may assume for our purposes that $z = (1, \dots, 1)$. Thus, we are interested in the orbit of the all-1s vector:
$$\mathcal{O} = \{\rho(\chi) : \chi \in A^*\} = \{(\chi(a_1), \dots, \chi(a_n)) : \chi \in A^*\}.$$

Definition 6.4. Let $I \subseteq R$ be an ideal. Le $R_{\le d}$ denote the vector space of polynomials in R of degree at most d, and let $I_{\le d}$ be the subspace $I \cap R_{\le d}$. The *affine Hilbert function* of R/I is $H\colon \mathbb{N} \to \mathbb{N}$, given by
$$H(d) := \dim_{\mathbb{C}} R_{\le d}/I_{\le d} = \dim_{\mathbb{C}} R_{\le d} - \dim_{\mathbb{C}} I_{\le d}.$$

Theorem 6.5. *Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and consider*

$$I = \{f \in R : f(\mathcal{O}) = 0\},$$

the ideal of polynomials vanishing on the orbit. Then

(1)
$$I = I(\Lambda) = \mathrm{Span}_{\mathbb{C}}\{x^u - x^v : u = v \bmod \Lambda\};$$

(2) *The affine Hilbert function of R/I is given by*
$$H(d) = \#\left\{\textstyle\sum_{i=1}^n n_i a_i : n_i \geq 0 \text{ for all } i \text{ and } \sum_i n_i \leq d\right\}.$$

Proof. This proof is due to the author and Donna Glassbrenner. It appears in [4]. Consider the matrix $M^{(d)}$ with rows indexed by A^* and columns indexed by the monomials of $R_{\leq d}$ (arranged in lexicographical order so that $M^{(d)}$ is naturally nested in $M^{(d+1)}$) given by

$$M^{(d)}_{\chi, x^u} = \prod_{i=1}^n \chi^{u_i}(a_i).$$

Recall the isomorphism

$$\begin{array}{ccc} A & \to & A^{**} \\ a & \mapsto & \bar{a} \end{array}$$

where $\bar{a}(\chi) := \chi(a)$. Thus, we can write

$$M^{(d)}_{\chi, x^u} = \prod_{i=1}^n \bar{a}_i^{u_i}(\chi) = \bar{a}^u(\chi)$$

where $\bar{a}^u := \prod_{i=1}^n \bar{a}_i^{u_i} \in A^{**}$. The x^u-th column of $M^{(d)}$ has entries $\bar{a}^u(\chi)$ as χ varies over A^*. In other words, it is the list of all values of the function $\bar{a}^u$. Thus, at least as far as linear algebra is concerned, the x^u-th column *is* $\bar{a}^u$. Since distinct characters are linearly independent, it follows that any linear dependence relations are the result of columns that are equal.

Now, the x^u-th and x^v-th columns of $M^{(d)}$ are equal exactly when $\bar{a}^u = \bar{a}^v$ are equal. This occurs exactly when $\sum_i u_i a_i = \sum_i v_i a_i$, which we write as $(u-v)\cdot a = 0$ where $a := (a_1, \ldots, a_n)$. In light of exact sequence (6.1), this condition is equivalent to $u - v \in \Lambda$.

A vector $(\alpha_u) \in \ker M^{(d)}$ if and only if

$$\sum_u \alpha_u \prod_{i=1}^n \chi^{u_i}(a_i) = 0$$

for all $\chi \in A^*$. Thus, $(\alpha_u) \in \ker M^{(d)}$ if and only if the polynomial $p = \sum_u \alpha_u x^u$ vanishes on $\mathcal{O}$, i.e., $p \in I$. Thus, elements of $I_{\leq d}$ correspond exactly with linear combinations among the columns of $M^{(d)}$. As these relations are due to equality among columns, as already noted, part 1 follows. For part 2, note that we have just shown that

$$\dim I_{\leq d} = \dim R_{\leq d} - \mathrm{rank}\ M^{(d)}.$$

Since distinct characters are linearly independent,

$$\mathrm{rank}\, M^{(d)} = \#\left\{\textstyle\sum_{i=1}^n n_i a_i : n_i \geq 0 \text{ for all } i \text{ and } \sum_i n_i \leq d\right\}.$$

□

Returning to the case of the toppling ideal, the exact sequence

$$0 \to \mathbb{Z}^n \xrightarrow{\widetilde{\Delta}} \mathbb{Z}^n \to \mathcal{S}(G) \to 0$$

has the form of exact sequence (6.1). The generators a_i are the configurations having exactly one grain of sand.

Corollary 6.6.

(1) *The toppling ideal is the set of polynomials vanishing on an orbit $\mathcal{O}$ of a faithful representation of $\mathcal{S}(G)^*$.*
(2) *The set of zeros of the toppling ideal is the finite set, $\mathcal{O}$. It thus has the symmetry of $\mathcal{S}(G)^*$, which is isomorphic to the sandpile group.*
(3) *If H_G is the affine Hilbert function for the toppling ideal, then $H_G(d)$ is the number of elements of $\mathbb{Z}^n/\widetilde{\mathcal{L}}$ represented by configurations containing at most d grains of sand. It is thus, the number of superstable configurations of degree at most d or, equivalently, the number of recurrent configurations c such that*
$$\deg(c) \geq \deg(c_{\max}) - d.$$

Proof. Part (1) follows directly from the first part of Theorem 6.5. For part (2), since $\mathcal{O}$ is a finite collection of points in $\mathbb{C}^n$, and $I(G) = I(\mathcal{O})$, it is a basic result of algebraic geometry that the set of zeros of $I(G)$ is $\mathcal{O}$. Part (3) is immediate from the second part of the theorem and the fact that r is recurrent if and only if $c_{\max} - r$ is superstable. □

Remark 6.7. Note that part (3) also follows directly from Theorem 5.12.

6.1.2. *Projective case.* An ideal J in $S = \mathbb{C}[x_1, \dots, x_{n+1}]$ is *homogeneous* if it has a set of homogeneous generators. The set of zeros of J is a subset of projective space:

$$Z(J) = \{p \in \mathbb{P}^n : f(p) = 0 \text{ for all homogeneous } f \in J\}.$$

The ring S/J is graded by the integers: $(S/J)_d := S_d/J_d$.

Definition 6.8. The *Hilbert function* of S/J is $H\colon \mathbb{N} \to \mathbb{N}$, given by

$$H(d) := \dim_{\mathbb{C}}(S/J)_d.$$

Continuing with the notation from 6.1.1, define the *homogenization* of Λ as

$$\Lambda^h := \left\{ \begin{pmatrix} \ell \\ -\deg(\ell) \end{pmatrix} \in \mathbb{Z}^{n+1} : \ell \in \Lambda \right\}.$$

Consider the exact sequence

$$0 \to \Lambda^h \to \mathbb{Z}^{n+1} \xrightarrow{M} A \oplus \mathbb{Z} \to 0.$$

where

$$M = \begin{pmatrix} a_1 & \dots & a_n & 0 \\ 1 & \dots & 1 & 1 \end{pmatrix}.$$

Apply $\mathrm{Hom}(\,\cdot\,, \mathbb{C}^\times)$ to get

$$\begin{array}{rcl} 1 \to A^* \times \mathbb{C}^\times & \to & (\mathbb{C}^\times)^{n+1} \to (\Lambda^h)^* \to 0 \\ (\chi, z) & \mapsto & (\chi(a_1)z, \dots, \chi(a_n)z, z), \end{array}$$

and the corresponding representation

$$\begin{array}{rcl} A^* \times \mathbb{C}^\times & \to & \mathrm{GL}(\mathbb{C}^{n+1}) \\ (\chi, z) & \mapsto & \mathrm{diag}(\chi(a_1)z, \dots, \chi(a_n)z, z). \end{array}$$

The orbit of $(1,\dots,1)$ under this representation is

$$\mathcal{O}^h = \{(\chi(a_1),\dots,\chi(a_n),1)\in\mathbb{P}^n : \chi\in A^*\}\subset\mathbb{P}^n.$$

Thus, $\mathcal{O}^h$ is the *projective closure* of the orbit $\mathcal{O}$ from the previous section.

Theorem 6.9. *Let $a^h = (a_1,\dots,a_n,0)$.*

(1) *The homogeneous ideal defining $\mathcal{O}^h$ is the lattice ideal for Λ^h, the homogenization of the lattice ideal for Λ:*

$$I^h = \{x^u - x^v : u = v \bmod \Lambda^h\}.$$

(2) *The Hilbert function for $\mathcal{O}^h$ (i.e., the Hilbert function of S/I^h) is*

$$H(d) = \#\{b\cdot a^h \in A : b\in\mathbb{N}^{n+1} \textit{ such that } \deg(b) = d\},$$

which is the same as the affine Hilbert function for $\mathcal{O}$.

Proof. Since $\mathcal{O}^h$ is the projective closure of $\mathcal{O}$, its ideal is I^h, the homogenization of the ideal defining $\mathcal{O}$, which is is clearly given by $\{x^u - x^v : u = v \bmod \Lambda^h\}$. The second part of the theorem follows from part 2 of Theorem 6.5 and the isomorphism of vector spaces

$$\begin{aligned} S_d &\to R_{\le d}\\ f &\mapsto f|_{x_{n+1}=1}, \end{aligned}$$

with inverse $g(x_1,\dots,x_n)\mapsto x_{n+1}^d\, g(x_1/x_{n+1},\dots,x_n/x_{n+1})$. □

Corollary 6.10. *Suppose $\Lambda = \widetilde{\mathcal{L}}$, the reduced Laplacian lattice of G, and that $\Delta(v_{n+1})\in\mathrm{Span}_{\mathbb{Z}}\{\Delta(v_i) : 1\le i\le n\}$ so that $\Lambda^h = \mathcal{L}$, the full Laplacian lattice (see the comments preceding Proposition 4.8).*

(1) *The homogenization of the toppling ideal is the ideal generated by all homogeneous polynomials vanishing on an orbit $\mathcal{O}^h$ of a faithful representation of $(\mathbb{Z}^{n+1}/\mathcal{L})^*$.*
(2) *The set of zeros of the homogenization of the toppling ideal is the finite set $\mathcal{O}^h$ having the symmetry of $\mathcal{S}(G)^*$.*

6.1.3. *The h-vector.* Let ΔH_G denote the first differences of the affine Hilbert function of a sandpile graph G. So $\Delta H_G(d) := H_G(d) - H_G(d-1)$. By Theorem 5.12, the value of $\Delta H_G(d)$ is the number of superstable configurations of degree d.

Definition 6.11. Let $h_d := \Delta H_G(d)$. The *postulation number* for G is the largest integer ℓ such that $h_\ell \neq 0$. The h-vector for G is $h = (h_0,\dots,h_\ell)$. The *Hilbert-Poincaré series* for G is $P_G(y) = \sum_{i=1}^{\ell} h_i y^i$.

Example 6.12. (*** Include the example of the genus two graph here? ***)

Let the vertices of G be $\{v_1,\dots,v_{n+1}\}$ with v_{n+1} as the sink, as usual. Let $I^h \subseteq I_h \subset S = \mathbb{C}[x_1,\dots,x_{n+1}]$ be the homogenization of the the toppling ideal and the homogeneous toppling ideal for G, respectively. These two ideals are identical when the hypothesis of Proposition 4.8 are satisfied. In any case, their zero-sets satisfy $Z(I^h)\supseteq Z(I_h)$. Pick a linear polynomial $f\in S$, that does not vanish at any point of $Z(I^h)$. For instance, we could take $f = x_i$ for any i. Multiplication by f gives rise to the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & S/I^h \arrow[r, "\cdot f"] \arrow[d, two heads] & S/I^h \arrow[r] \arrow[d, two heads] & S/(I^h + (f)) \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & S/I_h \arrow[r, "\cdot f"] & S/I_h \arrow[r] & S/(I_h + (f)) \arrow[r] & 0.
\end{tikzcd}$$

By this diagram and Theorem 6.9, we have the following relations among the first differences of Hilbert functions,

$$\begin{aligned} h_d = \Delta H_G(d) = \Delta H_{S/I^h}(d) &= H_{S/(I^h+(f))} \\ &\geq H_{S/(I_h+(f))} = \Delta H_{S/I_h}(d). \end{aligned} \tag{6.3}$$

6.1.4. *The Tutte polynomial.* Now let $G = (V, E)$ be any (weighted, directed) graph, and $e \in E$. Let $G - e$ denote the graph obtained from G by replacing $\mathrm{wt}(e)$ by $\mathrm{wt}(e) - 1$. In other words, imagine the endpoints e^- and e^+ attached by $\mathrm{wt}(e)$ edges, and remove one of these edges to obtain $G - e$. In particular, if $\mathrm{wt}(e) = 1$, this amounts to removing the edge e. Let G/e denote the graph obtained from G by identifying the endpoints of e and lowering the weight of e by one. We refer to these two operations on G as *deletion* and *contraction*. The edge e is called a *bridge* if $G - e$ has more components than G.

Definition 6.13. Let G be an undirected, weighted graph. Define the *Tutte polynomial*, $T_G(x, y)$ for G recursively, as follows. If E consists of i bridges, j loops, and no other edges, then

$$T_G(x, y) := x^i y^j.$$

In particular, $T_G = 1$ if G has no edges. Otherwise, if $e \in E$ is neither a bridge nor a loop, then

$$T_G := T_{G-e} + T_{G/e}.$$

It turns out the the Tutte polynomial is well-defined, independent of choices for deletions and contractions. It is well-known that

$$C_G(x) = (-1)^{\#V - \kappa(G)} x^{\kappa(G)} T_G(1 - x, 0),$$

where C_G is the chromatic polynomial of G and $\kappa(G)$ is the number of components of G. The following result relates another of specialization of the Tutte polynomial to the algebraic geometry of sandpiles.

Theorem 6.14 (Merino, [14])**.** *Let G be an undirected sandpile graph with postulation number ℓ. Then*

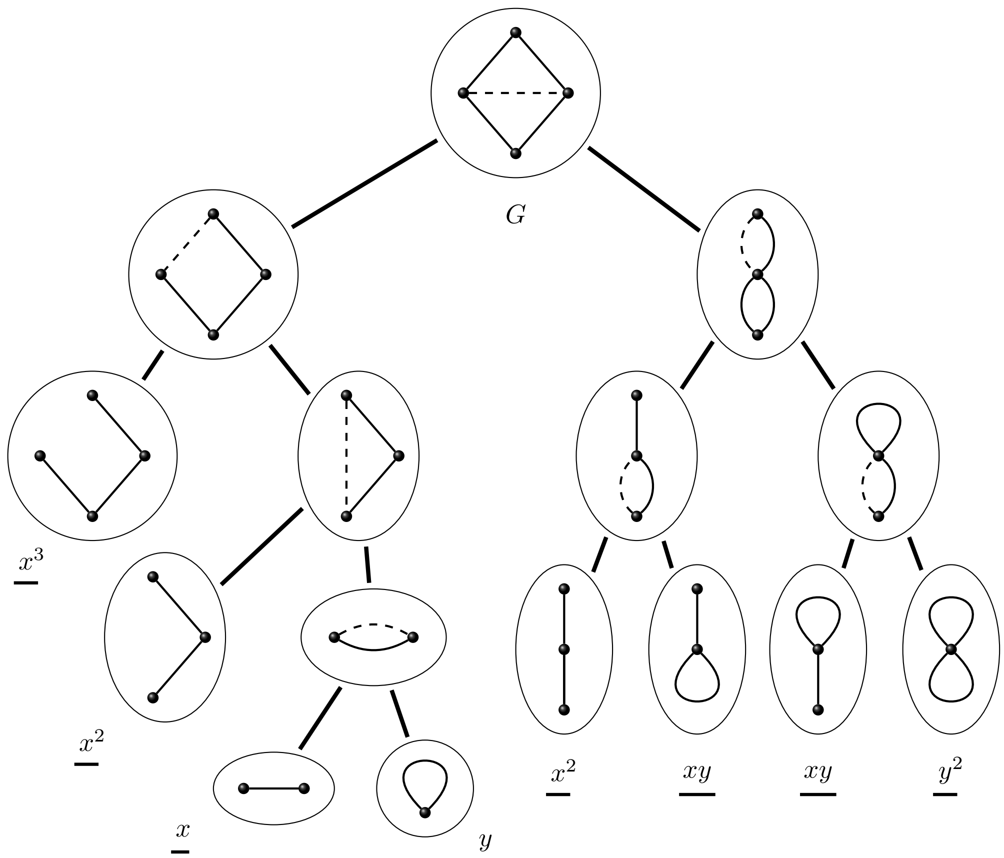
$$T_G(1, y) = \sum_{i=0}^{\ell} h_{\ell - i} y^i.$$

Corollary 6.15. *Let G be as in Theorem 6.14. Then*

1. *the Hilbert-Poincaré series for G is $y^\ell\, T(1, 1/y)$;*
2. *if d is the degree of the maximal stable configuration on G, then $y^{d-\ell}\, T(1, y)$ is the generating function for the recurrent configurations of G (by degree);*
3. *$T_G(1, 1)$ is the order of the sandpile group of G;*
4. *$T_G(0, 1)$ is the number of maximal superstable (or the number of minimal recurrent) configurations of G.*

Proof. These results follow immediately from Theorem 6.14. Part (2) uses the fact that c is superstable if and only if $c_{\max} - c$ is recurrent. $\square$

Example 6.16. Figure 6 shows the construction of the Tutte polynomial of a graph G. We have $T(1, y) = 4+3y+y^2$ and $T(1, 1) = 8$. Fixing the southern-most vertex of G as the sink gives a sandpile graph with h-vector $(1, 3, 4)$ and sandpile group of order 8.



$$T_G(x, y) = x + 2x^2 + x^3 + (1 + 2x)y + y^2$$

FIGURE 6. The Tutte polynomial of G.

6.1.5. *Cayley-Bacharach property.* Let $X \subset \mathbb{P}^n$ be a finite set of points in projective space, and let $I(X) \subset S := \mathbb{C}[x_1, \ldots, x_{n+1}]$ be the ideal generated by the homogeneous polynomials vanishing on X. If H_X is the Hilbert function of $S/I(X)$, then $H_X(d)$ is the number of linear conditions placed on the coefficients of a general homogeneous polynomial of degree d in S by requiring the polynomial vanish on the points of X. Thus, H_X is a monotonically increasing function which is eventually constant at $|X|$. The first value at which H_X takes the value $|X|$ is called the *postulation number* for X.

Definition 6.17. A finite set of points $X \subset \mathbb{P}^n$ is *Cayley-Bacharach* if it satisfies one of the following equivalent conditions.

(1) For each $p \in X$, and for each $d \in \mathbb{N}$.
$$H_{X\setminus\{p\}}(d) = \min\{H_X(d), |X| - 1\}.$$
(2) Every homogeneous polynomial with degree less than the postulation number for X and vanishing on all but one point of X must vanish on all of X.

Theorem 6.18. *The set of zeros of the homogeneous toppling ideal is Cayley-Bacharach.*

Proof. By Proposition 1.14 of [12], for any finite set of points, X, there is always at least on point p for which condition (1) of Defintion 6.17 holds. However, in our case, X is the orbit of a linear representation of the sandpile group. Thus, given any two points $p, q \in X$, there is a linear change of coordinates of $\mathbb{P}^n$ sending p to q. A linear change of coordinates does not change the Hilbert function. Hence, condition (1) holds for all points of X. □

Remark 6.19. Let X be the set of zeros of a homogeneous toppling ideal and define the first differences of its Hilbert function by $\Delta H_X(d) = H_X(d) - H_X(d-1)$ for all $d \in \mathbb{Z}$. It follows from results in [12] and the fact that X is Cayley-Bacharach, that if the last nonzero value of Δ_X is m, then there is a collection of m points $Y \subset X$ such that $X \setminus Y$ is Cayley-Bacharach. Moreover, if $m = 1$, then every subset of X of size $|X| - 1$ is Cayley-Bacharach.

7. Resolutions

In this section, we consider the minimal free resolution of the homogeneous toppling ideal, summarizing some of the results in [23]. First, we recall the language of divisors on graphs from [2] (extended to directed multigraphs). Let G be a directed multigraph as in §2. The free Abelian group, $\mathbb{Z}V$, on the vertices of G is denoted $\mathrm{div}(G)$, and its elements are called *divisors*. The *degree* a divisor $D = \sum_{v\in V} D_v\, v \in \mathrm{div}(G)$, is $\deg(D) := \sum_{v\in V} D_v$. A divisor is *principal* if it is in the Laplacian lattice, $\mathcal{L}$, defined in §2. Divisors D and D' are *linearly equivalent*, written $D \sim D'$, if $D - D'$ is principal. Note that linearly equivalent divisors must have the same degree. The group of divisors modulo linear equivalence is the *class group* of G, denoted $\mathrm{Cl}(G)$. We will usually denote a divisor class $[D] \in \mathrm{Cl}(G)$ by just D, choosing a representative divisor, when the context is clear. A divisor $D = \sum_{v\in V} D_v\, v$ is *effective*, written, $D \geq 0$ if $D_v \geq 0$ for all $v \in V$. The collection of all effective divisors linearly equivalent to D is called the *(complete) linear system* for D and denoted $|D|$. It only depends on the divisor class of D. The *support* of a divisor D is $\mathrm{supp}(D) := \{v \in V : D_v \neq 0\}$.

One might think of a divisor as an assignment of money to each vertex, with negative numbers denoting debt. Just as with configurations in the sandpile model, the Laplacian determines firing rules by which vertices can lend to or borrow from neighbors. Two divisors are linearly equivalent if one can be obtained from the other by a sequence of vertex lendings and borrowings. The complete linear system corresponding to a divisor is nonempty if there is a way for vertices to lend and borrow, resulting in no vertex being in debt.

7.1. **Riemann-Roch.** To recall the graph-theoretic Riemann-Roch theorem of [2], let $G = (V, E)$ be an undirected graph. Define the *genus* of G to be

$$g := \#E - \#V + 1.$$

Define the *dimension* of the linear system $|D$ for a divisor D on G to be

$$r(D) := \max\{k \in \mathbb{Z} : |D - E| \neq \emptyset \text{ for all } E \geq 0 \text{ with } \deg(E) = k\},$$

with $r(D) := -1$ if $|D| = \emptyset$. Note that $r(D)$ depends only on the divisor class of D. Define the *maximal stable divisor*,

$$D_{\max} := \sum_{v \in V} (\deg(v) - 1)v,$$

and the *canonical divisor*,

$$K := D_{\max} - \vec{1} = \sum_{v \in V} (\deg(v) - 2)v.$$

Theorem 7.1 (Riemann-Roch Theorem [2])**.** *Let G be an undirected graph. For all $D \in \operatorname{div}(G)$,*

$$r(D) - r(K - D) = \deg(D) + 1 - g.$$

7.2. **Resolutions and Betti numbers.** Let G be an arbitrary directed multigraph. Identify the vertices of G with the set $\{1, \ldots, n+1\}$, with $n+1$ being the sink. The polynomial ring, $S = \mathbb{C}[x_1, \ldots, x_{n+1}]$, is graded by the class group by letting the degree of a monomial x^D be $D \in \operatorname{Cl}(G)$. For each $D \in \operatorname{Cl}(G)$, let S_D be the $\mathbb{C}$-vector space generated by the monomials of degree D, and define the *twist*, $S(D)$, by letting $S(D)_F := S_{(D+F)}$ for each $F \in \operatorname{Cl}(G)$.

Let $I := I_h(G)$ be the homogeneous toppling ideal. A *free resolution* of I is an exact sequence

$$0 \leftarrow I \xleftarrow{\phi_0} F_1 \xleftarrow{\phi_1} F_2 \leftarrow \cdots \xleftarrow{\phi_r} F_r \leftarrow 0,$$

where each F_i is a free $\operatorname{Cl}(G)$-graded S-module, i.e.,

$$F_i = \bigoplus_{D \in \operatorname{Cl}(G)} S(-D)^{\beta_{i,D}}$$

for some nonnegative integers $\beta_{i,D}$, and where each ϕ preserves degrees. The *length* of the resolution is r. A free resolution is *minimal* if each of the $\beta_{i,D}$ is the minimum possible from among all free resolutions of I. In this case, the $\beta_{i,D}$ are called the *Betti numbers* of I. For instance, $\beta_{1,D}$ is the number of polynomials of degree D in a minimal generating set for I. We also define the i-th *coarsely graded Betti number* of I by $\beta_i = \sum_{D \in \operatorname{Cl}(D)} \beta_{i,D}$.

The following theorem states a well-known fact about resolutions of sets of points in projective space (the Cohen-Macaulay property).

Theorem 7.2. *The length of the minimal free resolution of the homogeneous toppling ideal is n, the number of nonsink vertices.*

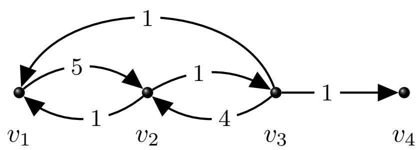


FIGURE 7. A Gorenstein sandpile graph G with sink v_4.

Example 7.3. Let G be as in Figure 7 and let $I = I(G)^h$. Then

$$0 \longleftarrow I \xleftarrow{\phi_0} \underset{\substack{0101\\0110\\1010\\1100\\0011}}{S^5} \xleftarrow{\phi_1} \underset{\substack{1011\\1101\\1110\\2010\\0111}}{S^5} \xleftarrow{\phi_2} \underset{1022}{S} \longleftarrow 0$$

is a minimal free resolution for I, where the ϕ_i are given by

$$\begin{aligned}
\phi_0 &= \begin{bmatrix} x_3^2 - x_2x_4 & x_2x_3 - x_1x_4 & x_2^2 - x_1x_3 & x_1x_2 - x_4^2 & x_1^2 - x_3x_4 \end{bmatrix}\\
\phi_1 &= \begin{bmatrix} x_2 & x_1 & 0 & x_4 & 0\\ -x_3 & -x_2 & x_1 & 0 & -x_4\\ x_4 & x_3 & 0 & x_1 & 0\\ 0 & 0 & -x_3 & -x_2 & x_1\\ 0 & 0 & x_4 & x_3 & -x_2 \end{bmatrix}\\
\phi_2 &= \begin{bmatrix} x_1^2 - x_3x_4\\ x_4^2 - x_1x_2\\ x_1x_3 - x_2^2\\ x_2x_3 - x_1x_4\\ x_3^2 - x_2x_4 \end{bmatrix}.
\end{aligned}$$

The grading of the S-modules is indicated below each of them. For example, the last S-module is $S(-(1,0,2,2))$.

The Betti numbers of I may be understood topologically. For $D \in \mathrm{Cl}(G)$, define the simplicial complex Δ_D on the vertices of G by $W \in \Delta_D$ if and only if $W \subseteq \mathrm{supp}(E)$ for some $E \in |D|$. The following version of Hochster's formula appeared as Lemma 2.1 of [17].

Theorem 7.4. *The Betti number $\beta_{i,D}$ is the dimension of the $(i-1)$-th reduced homology group $\widetilde{H}_{i-1}(\Delta_D;\mathbb{C})$ as a $\mathbb{C}$-vector space.*

Example 7.5. Let G again be as in Figure 7. For $D = v_1 + v_3 + v_4$, we saw in Example 7.3 that $\beta_{2,D} = 1$. We have

$$|D| = \{D, v_2 + 2v_3, 3v_1, 2v_2 + v_4\},$$

so the simplicial complex Δ_D is as pictured in Figure 8. Note $\dim_{\mathbb{C}} \widetilde{H}_2(\Delta_D;\mathbb{C}) = 1$, as asserted by Hochster's formula.

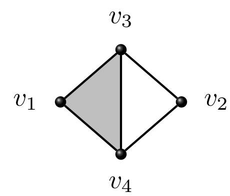


FIGURE 8. The simplicial complex Δ_D for Example 7.5.

7.3. **Minimal recurrents.** Again specialize to the case of an undirected graph G. As part of the Riemann-Roch theory, one defines the *non-special divisors* on G to be

$$\mathcal{N} := \{D \in \operatorname{div}(G) : \deg(D) = g - 1 \text{ and } |D| = \emptyset\}.$$

By the Riemann-Roch theorem, if $\deg(D) > g-1$, then $|D| \neq \emptyset$. So the nonspecial divisors are the divisors of maximal degree having empty linear system.

Fix $s \in V$ and consider the sandpile graph $G = (V, E, s)$. A recurrent configuration c on G is *minimal* if $c - v$ is not recurrent for any nonsink vertex v. It is well-known that (since G is undirected) the minimal recurrent configurations are exactly the recurrent configurations of minimal degree, namely of degree $\#E - \deg(s)$. (This result follows from Dhar's algorithm (cf. §2.4 and the proof of Theorem 8.27).) Similarly, one says that a superstable configuration c is a *maximal* if $c + v$ is not superstable for any nonsink vertex v. By Corollary 5.14, it follows that the maximal superstable configurations are exactly those of degree g.

We say that a divisor D on G is *unstable* if $D_v \geq \deg(v)$ for some $v \in V$ and that a divisor is *alive* if there is no stable divisor in $|D|$. Further, D is *minimally alive* if for all $v \in V$, we have that $D - v$. It is shown in [23] that a divisor D is alive if and only if $D \sim c + k\, c$ for some recurrent configuration c and some $k \geq \deg(s)$, and D is minimally alive if and only if $D \sim c + \deg(s)\, s$ for some minimal recurrent configuration c.

It is shown in [2] that a set of representatives for the distinct divisor classes of the non-special divisors is

$$\{c - s : c \text{ a maximal superstable configuration}\}.$$

If c is a maximal superstable configuration then

$$D_{\max} - (c - s) = (c_{\max} - c) + \deg(s)\, s.$$

Hence, $D_{\max} - (c - s)$ is minimally alive. Conversely, if c' is a minimal recurrent configuration, then $D' := c' + \deg(s)\, s$ is minimally alive and $D_{\max} - D'$ is non-special. Thus, on an undirected graph there is a bijective correspondence between: minimal recurrent configurations, maximal superstable configurations (hence, maximal parking functions), minimally alive divisors, and non-special divisors.

The following is Theorem 3.10 of [23]. The key ingredients in the proof are Hochster's formula and the Riemann-Roch theorem.

Theorem 7.6. *Let G be an undirected graph and $D \in \operatorname{div}(G)$. Let $r = \#V - 1$, the length of a minimal free resolution for G. Then the highest nonzero Betti number, β_r, is the number of minimal recurrent configurations on G. For $D \in \operatorname{div}(G)$,*

$$\beta_{r,D} \neq 0$$

if and only if D is minimally alive (in which case $\deg(D) = \#E$*).*

Example 7.7. We summarize many of the results of this paper using the graph G of genus $g = 2$ in Figure 9. The mathematical software Sage [21] was used for some of the calculations. The sandpile group for G is cyclic of order 8. Its toppling ideal

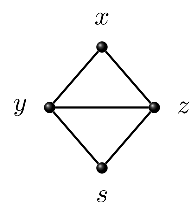


FIGURE 9. Genus two graph G.

is $I = (x^2 - yz, y^3 - xz, z^3 - xy, yz - 1)$, and its homogeneous toppling ideal is

$$I_h = I^h = (x^2 - yz, y^3 - xzs, z^3 - xys, yz - s^2, xz^2 - y^2 s, xy^2 - z^2 s).$$

Letting $\omega = \exp(2\pi/8)$, the zeros set of I is

$$Z(I) = \{((-1)^i, \omega^{-i}, \omega^i) : 0 \le i \le 7\} \subset \mathbb{C}^3,$$

which forms a cyclic group of order 8 under component-wise multiplication. With respect to the sandpile monomial ordering (`grevlex`) for which $x > y > z > s$, the normal basis for the coordinate ring of G is the spanned by 8 monomials:

$$R/I = \mathbb{C}[x, y, z]/I = \mathrm{Span}\{1, x, y, z, xy, xz, y^2, z^2\}.$$

The exponent vectors of the normal basis give the superstable configurations :

$$(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 2, 0), (0, 0, 2),$$

and dualizing, $c \to c_{\max} - c$, gives the recurrent configurations:

$$(1, 2, 2), (0, 2, 2), (1, 1, 2), (1, 2, 1), (0, 1, 2), (0, 2, 1), (1, 0, 2), (1, 2, 0).$$

(We use the notation $(c_1, c_2, c_3) := c_1\, x + c_2\, y + c_3\, z$, above.)

From the degrees of the monomials in the normal basis, one sees that the affine Hilbert function for G is

$$H_G(0) = 1, \quad H_G(1) = 3, \quad H_G(2)$$

with postulation number 2 (equal to g, the degree of the maximal superstables). The Tutte polynomial for G was calculated in Figure 6, and in accordance with Corollary 6.15, the Hilbert series for G is

$$y^2\, T_G(1, 1/y) = 1 + 3y + 4y^2.$$

The minimal free resolution for G is

$$\begin{array}{ccccccccc} 0 \longleftarrow & I & \xleftarrow{\phi_0} & S^6 & \xleftarrow{\phi_1} & S^9 & \xleftarrow{\phi_2} & S^4 & \longleftarrow 0 \\ & & & 0110 & & 0121 & & 0122 & \\ & & & 2000 & & 0211 & & 0212 & \\ & & & 0030 & & 1201{\times}2 & & 1022 & \\ & & & 1020 & & 1120 & & 1202 & \\ & & & 0300 & & 1021{\times}2 & & & \\ & & & 1200 & & 1210 & & & \\ & & & & & 0220 & & & \end{array}$$

The degrees are listed in x, y, z, s order. The $\mathrm{Cl}(G)$-degrees of the highest nonzero Betti numbers correspond to the minimal recurrent configurations as prescribed by Theorem 7.6. For instance, the degree 0122 corresponds to the minimal alive divisor $y + 2z + 2s$ and to the minimal recurrent configuration $(0, 1, 2)$. Thus, $\beta_2 = H_G(2)$, and the degrees of each of these divisors is $5 = \#E$.

As an example of Hochster's formula, let $D = 1021 = x + 2z + s$. The complete linear system for D is

$$|D| = \{1021, 2200, 0202, 0310\},$$

and Δ_D is the simplical complex pictured in Figure 10. We have

$$\beta_{2,1021} = \widetilde{H}_1(1021; \mathbb{C}) = 2.$$

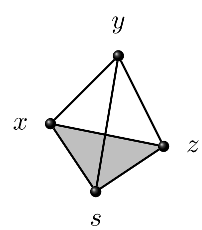


FIGURE 10. The simplicial complex Δ_D for Example 7.7.

7.4. **Conjecture.** Let $G = (V, E, s)$ be an undirected sandpile graph. For $U \subseteq V$, let $G|_U$ denote the subgraph of G induced by U, that is, the graph with vertices U and edges $e \in E$ such that both endpoints of e are in U. A *connected k-partition* of G is a partition $\Pi = \sqcup_{i=1}^k V_i$ of V such that $G|_{V_i}$ is connected for all i. The corresponding *k-partition graph*, G_Π, is the graph with vertices $\{V_1, \ldots, V_k\}$ and with edge weights

$$\mathrm{wt}(V_i, V_j) = \#\{e \in E : \text{one endpoint of } e \text{ is in } V_i \text{ and the other is in } V_j\}.$$

We consider G_Π to be a sandpile graph with sink vertex V_i where i is chosen so that $s \in V_i$.

The following conjecture appears as Corollary 3.29 in [23].

Conjecture 7.8. *Let $\mathcal{P}_k$ denote the set of connected k-partitions of G. Then*

$$\beta_k = \sum_{\Pi \in \mathcal{P}_{k+1}} \#\{c : c \textit{ a minimal recurrent configuration on } G_\Pi\}.$$

Example 7.9. Figure 11 displays the 5 connected 3-partitions of G along with their corresponding 3-partition graphs and h-vectors. The top value of each h-vector is the number of minimal recurrent configurations (or maximal superstable configurations) on the partition graph. Summing these top values gives β_2 for G.

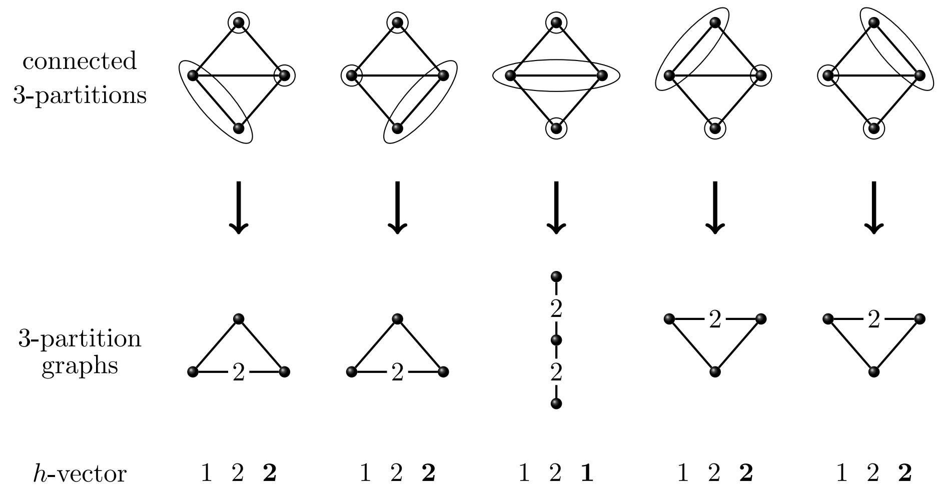


FIGURE 11. Second Betti number: $\beta_2 = 2 + 2 + 1 + 2 + 2 = 9$.

As a corollary to Conjecture 7.4, it is shown in [23] that

Corollary 7.10 (Conjecture)**.**

(1) *The number of polynomials, $\beta_!$, in a minimal generating set for the homogeneous toppling ideal of G is equal to the number of* cuts *(i.e., the number of connected 2-partitions) of G.*
(2) *For a tree on n vertices, $\beta_k = \binom{n-1}{k}$.*
(3) *The weight of an edge of G is change from one nonzero value to another, the β_k do not change.*
(4) *If G' is obtained from G by adding an edge to G (between two vertices of G), then $\beta_k(G) \leq \beta_k(G')$ for all k.*
(5) *For the complete graph on n vertices, K_n, we have that β_k is the number of strictly ascending chains of length k of nonempty subsets of $[n-1] := \{1, \ldots, n-1\}$.*

8. Gorenstein toppling ideals

This section characterizes toppling ideals that are complete intersection ideals and gives a method for constructing Gorenstein toppling ideals.

8.1. **Complete intersections.** If $V \subset \mathbb{P}^n$ is the solution set to a system of homogeneous polynomials, then V is a *complete intersection* if the ideal generated by all homogeneous polynomials vanishing on V can be generated by a set of polynomials with cardinality equal to the codimension of V in $\mathbb{P}^n$. Specializing to the case of sandpiles, we get the following definition.

Definition 8.1. Let $G = (V, E, s)$ be a sandpile graph with homogeneous toppling ideal I. Then G is a *complete intersection* sandpile graph if I is generated by $|V| - 1$ homogeneous polynomials. (We also say that I or the set of zeros of I is a complete intersection.)

Let $\mathcal{L}$ be a submodule of $\mathbb{Z}^{n+1}$ of rank n whose lattice ideal $I(\mathcal{L})$ is homogeneous. Then $I(\mathcal{L})$ is a *complete intersection* if it is generated by n homogeneous polynomials.

For the following, recall from §2 that sandpile has an *absolute* sink if its sink has outdegree 0.

Definition 8.2. For $i = 1, 2$, let $G_i = (V_i, E_i, s_i)$ be a sandpile graph with edge-weight function wt_i and absolute sink s_i. Suppose that the two graphs are vertex-disjoint. Let G be any graph with vertex set $V = V_1 \sqcup V_2$, and edge-weight function, wt, satisfying the following

(1) $\mathrm{wt}(e) = \mathrm{wt}_1(e)$ if $e \in E_1$,
(2) $\mathrm{wt}(e) = \mathrm{wt}_1(e)$ if $e \in E_1$,
(3) $\mathrm{wt}(u, v) = 0$ if $(u, v) \in (\widetilde{V}_1 \times V_2) \cup (V_2 \times V_1)$,
(4) $\mathrm{wt}(s_1, v) > 0$ for some $v \in V_2$.

We consider G to be a sandpile graph with s_2 as its absolute sink. Let $\Delta := \Delta_G$ be the Laplacian of G, and define

$$D := \Delta(s)|_{V_1} = \sum_{v \in V_1} \Delta(s)_v \, v,$$

a divisor on G_1. Then G is a *wiring of* G_1 *into* G_2 with *wiring divisor* D if $|D| \neq \emptyset$, i.e., if the complete linear system for D as a divisor on G_1 is nonempty (cf. §7).

Thus, to form a wiring of G_1 into G_2, one connects s_1 into G_2 with at least one edge and then adds edges from s_1 back into G_1 as determined by a divisor, D, on G_1 having a nonempty complete linear system. There always exists some wiring of G_1 into G_2. For instance, we could could take $D = k\, s_1$ for any $k > 0$ by connecting G_1 to G_2 with k edges from s_1 into G_2 (and no edges from s_1 back into G_1).

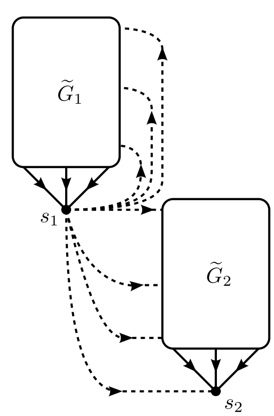


FIGURE 12. A wiring of G_1 into G_2.

Notation 8.3. For any sandpile graph $G = (V, E, s)$, with Laplacian Δ_G, we let $\Delta_G^\circ = \Delta_G|_{\widetilde{V}}$. Thus, $\Delta_G^\circ : \mathbb{Z}\widetilde{V} \to \mathbb{Z}V$, and in terms of matrices, Δ_G° is formed from Δ_G by removing the column corresponding to the sink—a column of zeros if G has an absolute sink. We will call Δ_G° the *restricted Laplacian* of G.

With this notation, if G is a wiring of G_1 into G_2, then

$$\Delta_G^\circ = \begin{pmatrix} \Delta_{G_1}^\circ & 0 & \alpha \\ 0 & \Delta_{G_2}^\circ & \beta \end{pmatrix}$$

where exactly one entry of α is positive (corresponding to s_1) and $\beta \leq 0$. The last column corresponds to s_1, and the wiring divisor is $D = \alpha$.

If G_1 is a single point with no edges, then we regard $\Delta_{G_1}^\circ$ as the 1×0 empty matrix, and α will be a single integer, as in the following example.

Example 8.4. Let G_1 be the graph with a single vertex s_1 and no edges. Let G_2 have vertex set $\{v_2, v_3, s_2\}$ and edge set $\{(v_2, s_2), (v_3, s_2)\}$. Figure 13 illustrates a wiring, G, of G_1 into G_2. The wiring divisor is $D = 2s_1$. The restricted Laplacian

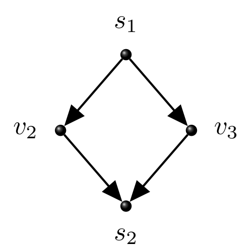


FIGURE 13. A wiring of G_1 into G_2.

of G is, with respect to the indicated vertex ordering,

$$\Delta_G^\circ = \begin{array}{c} \\ s_1 \\ v_2 \\ v_3 \\ s_2 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} v_2 & v_3 & s_1 \end{array} \\ \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 0 \end{pmatrix} \end{array}.$$

Definition 8.5. A directed multigraph G is *completely wired* if it is a single vertex with no edges or if it is the wiring of one completely wired graph into another.

Example 8.6. Every directed acyclic graph is completely wired.

Definition 8.7. An integral matrix is *mixed* if each column contains both positive and negative entries. An integral matrix is *mixed dominating* if it does not contain a mixed square submatrix.

Empty $d \times 0$ matrices are mixed dominating by convention. The following two theorems are established in [16], [10].

Theorem 8.8. *Let $\mathcal{L}$ be a submodule of $\mathbb{Z}^{n+1}$ of rank n such that the associated lattice ideal $I(\mathcal{L})$ is homogeneous. Then $I(\mathcal{L})$ is a complete intersection if and only if there exists a basis $u_1, \ldots, u_n$ for $\mathcal{L}$ such that the matrix whose columns are the u_i is mixed dominating.*

Theorem 8.9. *If M is a mixed dominating matrix, then by reordering its columns and rows we may obtain*

$$M' = \left(\begin{array}{cc|c} M_1 & 0 & \alpha \\ 0 & M_2 & \beta \end{array}\right)$$

where the M_i are mixed dominating, $\alpha \geq 0$, and $\beta \leq 0$.

It is allowable for the matrix M_1 in Theorem 8.9 to be the empty $d \times 0$ matrix, in which case we would have

$$M' = \left(\begin{array}{c|c} 0 & \alpha \\ M_2 & \beta \end{array}\right),$$

where the upper-left block is a zero matrix with d rows. A similar statement holds if M_2 is the $d \times 0$ matrix, in which case we would have a lower-left zero matrix block.

We now characterize complete intersection sandpile graphs.

Theorem 8.10. *Let $\mathcal{L}$ be a submodule of $\mathbb{Z}^{n+1}$ of rank n such that the associated lattice ideal $I(\mathcal{L})$ is a complete intersection. Then there exists a completely wired graph G whose Laplacian lattice is $\mathcal{L}$, and hence, $I(\mathcal{L}) = I(G)^h$ where $I(G)^h$ is the homogeneous toppling ideal of G.*

Proof. We proceed by induction, the case $n = 0$ being trivial. Let $u_1, \ldots, u_n$ be a basis for $\mathcal{L}$, and let M be the matrix whose columns are the u_i. By assumption, $\deg(u_i) = 0$ for all i. (Here, $\deg(u_i)$ denotes the degree of u_i as a divisor, i.e., the sum of the components of u_i.) By Theorems 8.8 and 8.9 we may assume that

$$M = \left(\begin{array}{cc|c} M_1 & 0 & \alpha \\ 0 & M_2 & \beta \end{array}\right)$$

where the M_i are mixed dominating, $\alpha \geq 0$, and $\beta \leq 0$. Each column of M_1 and M_2 has entries that sum to zero. By our rank assumption, it follows that M_1 and M_2 are matrices of full rank, each with one more row than column. By induction, there exist completely wired graphs G_1 and G_2 such that $\operatorname{im}(\Delta_{G_i}) = \operatorname{im}(M_i)$ for $i = 1, 2$. Let s_1 be the sink of G_1. Let c be any nonnegative configuration on G_1 with full support and contained in $\operatorname{im}(\widetilde{\Delta}_{G_1})$, the reduced Laplacian lattice for G_1. For instance, we could take $c = \delta - \delta^\circ$ where $\delta = \sum_{v \in \widetilde{V}_1} (\operatorname{outdeg}(v) + 1)\, v$. Define the divisor $D = c - \deg(c)\, s_1 \in \operatorname{im}(\Delta_{G_1}) = \operatorname{im}(\Delta^\circ_{G_1})$. Take $k \in \mathbb{N}$ such that $k \cdot c + \deg(\alpha)\, s_1 \geq \alpha$. Now

$$M' = \left(\begin{array}{cc|c} \Delta^\circ_{G_1} & 0 & \alpha - kD \\ 0 & \Delta^\circ_{G_2} & \beta \end{array}\right)$$

has the same column span as M, and $M' = \Delta^\circ_G$ where G is the wiring of G_1 into G_2 with wiring divisor $\alpha - kD$. Then G is completely wired and, up to an ordering of its vertices, its full Laplacian lattice is $\mathcal{L}$. $\square$

Example 8.11. The graph of Example 4.9 is a complete intersection sandpile graph. It is not completely wired, but its Laplacian lattice is the same as that for the completely wired graph consisting of a single directed edge connecting v_1 to v_2.

Theorem 8.12. *If the graph G is completely wired, then $I(G)^h$ is a complete intersection.*

Proof. If G has only one vertex, then $I(G) = \{0\}$ is a complete intersection, so we will again proceed by induction, now on $|V(G)|$. Assume $|V(G)| > 1$, and that G is the wiring of some graph G_1 with sink s into another graph G_2, with wiring divisor D. Let $\beta = \Delta_G(s)|_{V_2}$. Then

$$\Delta_G^\circ = \left(\begin{array}{cc|c} \Delta_{G_1}^\circ & 0 & D \\ 0 & \Delta_{G_2}^\circ & \beta \end{array} \right).$$

By Theorem 8.8 and induction, there exist M_1 and M_2 with $\mathrm{im}(M_i) = \mathrm{im}(\Delta_{G_i})$ for $i = 1, 2$, and $E \in |D|$, such that

$$M = \left(\begin{array}{cc|c} M_1 & 0 & E \\ 0 & M_2 & \beta \end{array} \right)$$

has the same column span as Δ_G° and is mixed dominating. So $I(G)^h$ is a complete intersection by Theorem 8.8. □

8.2. **Gorenstein sandpile graphs.** Having characterized complete intersection sandpile graphs, we proceed to give a method for constructing sandpile graphs with Gorenstein toppling ideals. Our basic reference for Gorenstein ideals is [11].

Notation 8.13. Let $S = \mathbb{C}[x_1, \ldots, x_{n+1}]$, and let I be a homogeneous ideal in S. Let S_d be the $\mathbb{C}$-vector space generated by all homogeneous polynomials of degree d, and let $I_d := I \cap S_d$. Define $A = S/I$, and let $A_d := (S/I)_d := S_d/I_d$. Let

$$\mathfrak{m} = (x_1, \ldots, x_{n+1})$$

denote the unique maximal homogeneous ideal in either S or in A.

Definition 8.14. The *socle* of A is

$$\mathrm{Soc}(A) := (0 : \mathfrak{m}) := \{f \in A : f\,\mathfrak{m} = 0\}.$$

Definition 8.15. The ring A is Artinian if $\dim_{\mathbb{C}} A < \infty$. In that case, we write

$$A = \mathbb{C} \oplus A_1 \oplus \cdots \oplus A_\ell,$$

with $A_\ell \neq 0$. The number ℓ is the *socle degree* of A. It is the least number ℓ such that $\mathfrak{m}^{\ell+1} \subseteq I$.

Definition 8.16. The ring A is *Gorenstein* if it is Artinian and $\dim_{\mathbb{C}} \mathrm{Soc}(A) = 1$ (so $\mathrm{Soc}(A) = A_\ell$ and $\dim_{\mathbb{C}} A_\ell = 1$).

Proposition 8.17. *Suppose A is Artinian with socle degree ℓ. Then A is Gorenstein if and only if $\dim_{\mathbb{C}} A_\ell = 1$ and the pairing given by multiplication*

$$A_d \times A_{\ell-d} \to A_\ell \approx \mathbb{C}$$

is a perfect pairing.

Proof. See the proof of, and remarks following, Proposition 8.6, [11]. □

As an easy corollary, we have

Corollary 8.18. *The Hilbert function of an Artinian Gorenstein ring A is symmetric. That is, if the socle degree of A is ℓ, then*

$$H_A(d) = H_A(\ell - d),$$

for all d.

Now let $S' = \mathbb{C}[y_1, \ldots, y_{n+1}]$, and let S act on S' by treating each x_i as the differential operator $\partial/\partial y_i$.

Theorem 8.19. *(Macaulay, cf. Theorem 8.7* [11]*) The ring $A = S/I$ is Gorenstein with socle degree ℓ if and only if there exists a nonzero $g \in S'_\ell$ such that*

$$I = \operatorname{ann}(g) := \{f \in S : f(\partial/\partial y_1, \ldots, \partial/\partial y_{n+1})\, g = 0\}.$$

Now consider the case where I is the homogeneous toppling ideal for a sandpile graph G with vertices $\{v_1, \ldots, v_{n+1}\}$. Let $X = Z(I)$ be the zero set of I as discussed in section 6.1.2. Let $a \in S$ be a linear polynomial that does not vanish at any point of X. For instance, a may be any of the indeterminates, x_i. Restricting the exact sequence given by multiplication by a,

$$0 \to A \xrightarrow{\cdot a} A \to A/(a) \to 0,$$

to each degree d, we find that the Hilbert function for $A/(a)$ is the first differences of the Hilbert function for A, i.e., $H_{A/(a)}(d) = \Delta H_A(d)$. It then follows from (6.3) that $A/(a)$ is Artinian.

Definition 8.20. Continuing the notation from above, the ring $A/(a)$ is called an *Artinian reduction* of A. Let ℓ be the socle degree of an Artinian reduction of A, and let $h_d = \Delta H_A(d)$ for $d = 0, \ldots, \ell$. Then $(h_0, \ldots, h_\ell)$ is the *homogeneous h-vector* of G (or I or X).

Remark 8.21. The homogeneous h-vector and the h-vector appearing in Definition 6.11 are identical in the case the $\Delta(v_{n+1}))$ is in the span of $\{\Delta(v_i) : 1 \leq i \leq n\}$ (see the discusion after Example 6.12).

Definition 8.22. We say G is a *Gorenstein sandpile graph* if its homogeneous coordinate ring has a Gorenstein Artinian reduction. We also say that I and X are (arithmetically) Gorenstein.

Remark 8.23.

1. Using the notation preceding Definition 8.20, it turns out that if A has a Gorenstein Artinian reduction, then every Artinian reduction of A is Gorenstein.
2. The notion of a Gorenstein ideal is much more general, but requires a discussion of the Cohen-Macaulay property, which our toppling ideals (defining a finite set of projective points) satisfy automatically, (cf. [9]).

It is well-known that complete intersections ideals are Gorenstein (cf. §21.8[9]). In particular, we have the following.

Theorem 8.24. *Let G be a sandpile graph. If G is a complete intersection, then G is Gorenstein.*

Theorem 8.25. *Let I be the homogeneous toppling ideal of the sandpile graph G having $n+1$ vertices. The following are equivalent:*

1. *G is Gorenstein.*
2. *If the minimal free resolution for I is*
$$0 \leftarrow I \xleftarrow{\phi_0} F_0 \xleftarrow{\phi_1} F_1 \leftarrow \cdots \xleftarrow{\phi_n} F_n \leftarrow 0,$$
then $F_n \approx S$ as an S-module.
3. *The homogeneous h-vector for G is symmetric.*

Proof. The equivalence of items (1) and (2) is a standard result (cf. [9]). The equivalence of items (1) and (3) follows by [8] since I is a Cayley-Bacharach ideal by Theorem 6.18. □

Example 8.26. Let G be as in example 7.3. We saw that the last nonzero module in the free resolution for $I(G)^h$ is $S(-(1,0,2,2))$, which is isomorphic to S as an S-module. Thus, the caption for Figure 7, stating that G is Gorenstein, is justified by (2) above.

Define a *loopy tree* to be a (finite) graph that is formed from a weighted, undirected tree by adding weighted loops at some (maybe none) of the vertices.

Theorem 8.27. *For an undirected sandpile graph G, the following are equivalent:*

(1) *G is a loopy tree,*
(2) *G is a complete intersection,*
(3) *G is Gorenstein.*

Proof. Let $G = (V, E, s)$ be a undirected sandpile graph. First suppose that G is a loopy tree. Removing any outgoing edges from s leaves a completely wired graph having the same homogeneous toppling ideal as G. Hence, G is a complete intersection by Theorem 8.12, and hence G is Gorenstein by Theorem 8.24.

We now assume that G is not a loopy tree. Since the lattice ideal of G is not affected by loops, for ease of exposition we assume that G has no loops. By Theorem 7.6 and Theorem 8.25 (2), we have that G is Gorenstein if and only if it has a unique minimal recurrent configuration.

To characterize the minimal recurrent configurations, let $\prec$ be a total ordering of the vertices such that for all nonsink vertices v, (i) $s \prec v$, and (ii) there exists $u \prec v$ such that $\{u, v\} \in E$. Define the configuration $c_\prec$ by

$$c_{\prec,v} := \deg(v) - \#\{v \in V : \{U, v\} \in E \text{ and } u \prec v\}.$$

We now invoke Dhar's burning algorithm. Let b be the minimal burning configuration for G. By Theorem 2.26 it has script $\vec{1}$, and by Theorem 2.25, a configuration c is recurrent if and only if each nonsink vertex fires in the stabilization of $b+c$. Note that $b+c$ is obtained by starting with c and firing the sink vertex. It follows that $c_\prec$ is a minimal recurrent configuration and that all minimal recurrent configurations arise as $c_\prec$ for some ordering $\prec$ satisfying (i) and (ii), above.

Let C be a (undirected) cycle in G. Choose a path P in G starting at s and going to a vertex of C, then traveling around C. To be precise, let $u_1, \dots, u_i$ be distinct vertices forming a path in G (so $\{u_\ell, u_{\ell+1}\} \in V$ for all ℓ) with $u_1 = s$ and u_i a vertex in C. Assume that u_i is the first vertex in the path to be in C. (If s is in C, then $i = 1$.) Next, let $u_i, \dots, u_{i+j}$ be the vertices in the cycle C, in order. Then P be the path $u_1, \dots, u_{i+j}$. Let $\prec_1$ be any total ordering satisfying (i) and (ii), above, with

$$u_1 \prec_1 \cdots \prec_1 u_{i+j},$$

and such that $u_k \prec_1 v$ for all u_k and all vertices v not in P. Let $\prec_2$ be any total order satisfying (i) and (ii) with

$$u_1 \prec_2 \cdots \prec_2 u_i \prec_2 u_{i+j} \prec_2 u_{i+1} \prec_2 u_{i+2} \prec_2 \cdots \prec_2 u_{i+j-1},$$

and such that $u_k \prec_2 v$ for all u_k and all vertices v not in P. It follows that $c_{\prec_1}$ and $c_{\prec_2}$ are distinct minimal recurrent configurations on G. Hence, G is not Gorenstein. □

(*** fix the following ***) By Theorem 5.12, an Artinian reduction of A has the set

$$\{x^c : c \text{ is a superstable configuration of } G\}$$

as a normal basis. It follows that the socle degree ℓ of A is the maximum of the degrees of the superstable configurations of G. Hence, by Proposition 8.17, a sandpile graph is Gorenstein if and only if there exists a bijection between the superstable configurations of degree k and those of degree $\ell - k$.

Lemma 8.28. *Let G_1 be the graph on a single vertex v and let G_2 be a Gorenstein sandpile graph. Let G be a wiring of G_1 into G_2. Then G is Gorenstein.*

Proof. Let A be the set of superstable configurations on G_2 and let $\ell = \max\{\deg(a) : a \in A\}$. Let $f\colon A \to A$ be a bijection such that $\deg(f(a)) = \ell - \deg(a)$ for all $a \in A$. Le $d := \mathrm{outdeg}(v)$. Since there are no edges from vertices of G_2 to v in G, the set of superstable configurations on G is

$$B := \{kv + a : a \in A \text{ and } 0 \le k < d\}.$$

Let $m := \max\{\deg(b) : b \in B\} = \ell + d - 1$. Define $g\colon B \to B$ by $g(kv + a) = (d - 1 - k)v + f(a)$ where $a \in A$. Then g is a bijection and

$$\deg(g(kv + a)) = \deg((d - k - 1)v + f(a)) = \ell + d - 1 - \deg(kv + a).$$

It follows that G is Gorenstein. □

Lemma 8.29. *Let G_1 be a Gorenstein sandpile graph with absolute sink s and let G be a wiring of G_1 into the graph on a single vertex v with no edges. Then G is Gorenstein.*

Proof. Let $\Delta = \Delta_G$ be the Laplacian matrix for G, and let D be the wiring divisor of G. If d is the weight of the edge from s to v, then

$$\Delta_G^\circ = \begin{pmatrix} \Delta_{G_1}^\circ & D \\ 0 & -d \end{pmatrix}.$$

Since $|D| \neq \varnothing$ by the definition of a wiring, there exists some effective divisor $E \sim_{G_1} D$. Thus, we can replace the last column of Δ_G° with

$$\begin{pmatrix} E \\ -d \end{pmatrix}$$

without changing the column span, and hence without changing the associated lattice ideal. Negating this column and swapping rows, the matrix Δ_G° becomes

$$\Delta^\circ := \begin{pmatrix} 0 & d \\ \Delta_{G_1}^\circ & -E \end{pmatrix},$$

which is the restricted Laplacian for a wiring of vertex v into G_1. This graph is Gorenstein by Lemma 8.28. □

Theorem 8.30. *Let G_1 and G_2 be Gorenstein sandpile graphs with absolute sinks, and let s be the sink of G_1. If G is a wiring of G_1 into G_2, then G is Gorenstein.*

Proof. Let D be the wiring divisor of G. Let G' be the wiring of G_1 into the graph on the single vertex s with wiring divisor D. Let A' be the set of superstable configurations on G' and let $\ell' = \max\{\deg(c) : c \in A'\}$. Since G' is Gorenstein by Lemma 8.29, there exists a bijection $f'\colon A' \to A'$ such that $\deg(f'(c)) = \ell' - \deg(c)$.

Let A_2 be the set of superstables on G_2, let $\ell_2 = \max\{\deg(c) : c \in A_2\}$, and let $f_2 \colon A_2 \to A_2$ be a bijection such that $\deg(f_2(c)) = \ell_2 - \deg(c)$.

Clearly, if c is superstable on G, then $c|_{\widetilde{V}_2} \in A_2$, and $c|_{V_1} \in A'$. Conversely, if $c' \in A'$ and $c_2 \in A_2$, then the configuration $c' + c_2$ is superstable on G. Let $A = \{c' + c_2 : c' \in A', c_2 \in A_2\}$, so that A is the set of superstable configurations on G, and $\max\{\deg(c) : c \in A\} = \ell' + \ell_2 =: \ell$. Define the function $f \colon A \to A$ by $f(c' + c_2) = f'(c') + f_2(c_2)$, where $c' \in A'$ and $c_2 \in A_2$. Then f is a bijection, and $\deg(f(c' + c_2)) = \ell - \deg(c' + c_2)$. Hence, G is Gorenstein. □

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