Let R be a commutative ring with identity, 1. An *R-module* is a commutative group M together scalar multiplication, i.e., an mapping $\cdot\colon R \times M \to R$ satisfying:

1. $(r+s)m = rm + sm$ for all $r, s \in R$ and $m \in M$
2. $r(m+n) = rm + rn$ for all $r \in R$ and $m, n \in M$
3. $(rs)m = r(sm)$ for all $r, s \in R$ and $m \in M$
4. $1m = m$ for all $m \in M$.

A subgroup N of an R-module M is a *submodule* if $rn \in N$ for all $r \in R$ and $n \in N$. In that case, we can form the *quotient module* M/N consisting of the group of cosets $m + N$ for $m \in M$, with the obvious R-module structure, $r(m+N) = rm + N$. A *homomorphism of R-modules* M and N is a group homomorphism $f\colon M \to N$ satisfying $f(rm) = rf(m)$ for all $r \in R$ and $m \in M$. Two R-modules M, N are *isomorphic* if there are R-module homomorphisms $f\colon M \to N$ and $g\colon N \to M$ that are inverses of each other.

Examples

1. A $\mathbb{Z}$-module is just an abelian group.
2. If R is a field, an R-module is exactly a vector space over R.
3. For any ring R, every ideal of R is an R-submodule.
4. If $f\colon M \to N$ is a homomorphism of R-modules, then
   (a) the *kernel* of f is the R-submodule of M, $\ker(f) := f^{-1}(0)$
   (b) the *image* of f is the R-submodule of N, $\mathrm{im}(f) := f(M)$
   (c) the *cokernel* of f is the R-module $N/\mathrm{im}(f)$.
   (d) the mapping f is injective iff $\ker(f) = 0$ and is surjective iff $\mathrm{coker}(f) = 0$; it is an isomorphism iff it is injective and surjective.

A sequence of R-module homomorphisms

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

is *exact* (or exact at M) if $\mathrm{im}(f) = \ker(g)$. A *short exact sequence* of R-modules is a sequence of R-module homomorphisms

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

exact at M', M, and M''.

Exercise 1 For a short exact sequence of R-modules as above

1. f is injective
2. g is surjective
3. M'' is isomorphic to $\operatorname{coker}(f)$.
4. If R is a field, so that M', M, and M'' are vector spaces, then

$$\dim M = \dim M' + \dim M''.$$

In general, as sequence of R-module mappings

$$\cdots \to M_i \to M_{i+1} \to \dots$$

is *exact* if it is exact at each M_i (except the first and last, if they exist).

Consider a commutative diagram of R-modules with exact rows

$$\begin{tikzcd}
 & M' \arrow[r, "f"] \arrow[d, "\phi'"] & M \arrow[r, "g"] \arrow[d, "\phi"] & M'' \arrow[r] \arrow[d, "\phi''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "h"] & N \arrow[r, "k"] & N'' &
\end{tikzcd}$$

The *snake lemma* says there is an exact sequence

$$\ker \phi' \to \ker \phi \to \ker \phi'' \to \operatorname{coker} \phi' \to \operatorname{coker} \phi \to \operatorname{coker} \phi''$$

If f is injective, then so is $\ker \phi' \to \ker \phi$, and if k is surjective, so is $\operatorname{coker} \phi \to \operatorname{coker} \phi''$.

Exercise 2 Prove the snake lemma.