Definition 1 Let Γ be a directed graph, and let s be a vertex of Γ. A *spanning tree of* Γ *directed into* s, also called a *spanning tree rooted at* s, is a connected subgraph of Γ containing all of the vertices and such that each vertex has outdegree 1, except for s, which has outdegree 0.

Let K be the complete digraph on $n+1$ vertices $V = \{v_0, \ldots, v_n\}$. The edge set for K is $V \times V$. In particular, K has a loop at each vertex. For each directed edge (v_i, v_j) of K assign an indeterminate x_{ij}. Define the $(n+1) \times (n+1)$ matrix L by

$$L_{ij} = \begin{cases} \sum_{k \neq i} x_{ik} & \text{for } i = j \\ -x_{ij} & \text{otherwise,} \end{cases}$$

for $i, j \in \{0, 1, \ldots, n\}$. Thus, the sum of the columns is the zero vector, and row i encodes information about the edges leaving vertex v_i. Let $L^{(k)}$ denote the $n \times n$ submatrix formed by deleting the k-th row and column of L. For each k, let T_k denote the collection of all directed spanning trees of K rooted into v_k. Thus, if $\tau \in T_k$, then for each vertex v_i there is unique a directed path in τ to v_k. Define the *weight* of $\tau \in T_k$, denoted $\mathrm{wt}(\tau)$, to be the product of all x_{ij} such that (v_i, v_j) is an edge in τ.

Theorem 2 *The determinant of* $L^{(k)}$ *is the sum of the weights of all spanning trees directed into* v_k*:*

$$\det L^{(k)} = \sum_{\tau \in T_k} \mathrm{wt}(\tau).$$

Proof. Since a permutation of the vertices induces a corresponding permutation of the rows and columns of L, it suffices to consider the case $k = 0$. Let Υ be the collection of all directed graphs on the vertices $v_0, \ldots, v_n$ in which the outdegree of v_0 is 0 and the outdegree of all other vertices is 1. Starting at any vertex of such a graph and following outgoing edges either leads to a cycle or to v_0. In this way, there is a connected component associated with each cycle and one connected component which is a directed tree rooted into v_0. The cycles of $\gamma \in \Upsilon$ will be identified with elements of the symmetric group S_n: a cycle with vertices $v_{i_1}, \dots, v_{i_k}$, listed in order, is identified with the permutation $(i_1, \dots, i_k)$.

The elements of Υ are in one-to-one correspondence with monomials of the form $x_{1i_1} \dots x_{ni_n}$ with $i_1, \dots, i_n \in \{0, 1, \dots, n\}$ by letting $\gamma \in \Upsilon$ correspond to

$$x_\gamma := \prod_{(v_i,v_j)\in E(\gamma)} x_{ij} = x_{1i_1} x_{2i_2} \dots x_{ni_n}.$$

where $E(\gamma)$ denotes the edges of γ.

We can compute the determinant by the formula

$$\det L^{(0)} = \sum_{\pi\in S_n} \operatorname{sgn}(\pi) L_{1\pi(1)} \dots L_{n\pi(n)}. \tag{1}$$

If $i \notin \operatorname{Fix}(\pi)$, then $L_{i\pi(i)} = -x_{i\pi(i)}$ and otherwise, $L_{i\pi(i)} = L_{ii} = \sum_{k\neq i} x_{ik}$. The monomials in the x_{ij} appearing in the expansion of $L_{1\pi(1)} \dots L_{n\pi(n)}$ are exactly those x_γ where the graph $\gamma \in \Upsilon$ includes the nontrivial cycles of π (and possibly more cycles).

For $\gamma \in \Upsilon$, what is the coefficient of x_γ in the expansion of (1)? We get a contribution $\pm x_\gamma$ from the expansion of each summand of (1) corresponding to a permutation π whose cycles are contained in the collection of cycles of γ. For such a π consider $\operatorname{sgn}(\pi) L_{1\pi(1)} \dots L_{n\pi(n)}$. To form x_γ we need to choose an indeterminate from each factor $L_{i\pi(i)}$. If $i \in \operatorname{Fix}(\pi)$, then the choice made does not affect the sign. Group the remaining factors according to the cycles of π. If $(i_1, \dots, i_k)$ is a cycle of π, then $L_{i_1i_2} L_{i_2i_3} \dots L_{i_ki_1} = (-1)^k x_{i_1i_2} x_{i_2i_3} \dots x_{i_ki_1}$. Thus, a factor of -1 is contributed by each even cycle. On the other hand, each odd cycle contributes a factor of -1 when calculating $\operatorname{sgn}(\pi)$. Hence, if π is the product of t disjoint cycles of γ, then the corresponding summand of (1) contributes $(-1)^t$ to the coefficient of x_γ. In this way, if γ has $s > 0$ cycles, the coefficient of x_γ is seen to be

$$\sum_{t=0}^{s} \binom{s}{t} (-1)^t = (1-1)^s = 0.$$

That leaves the case where γ has 0 cycles, i.e., γ is a directed spanning tree directed into v_0. In that case x_γ must come from the summand of (1) corresponding to the identity permutation:

$$L_{11} \dots L_{nn} = (\sum_{k_1\neq 1} x_{1k_1}) \dots (\sum_{k_n\neq n} x_{nk_n}).$$

Each such x_γ arises uniquely in the expansion. $\square$

The theorem specializes to the case of any finite, weighted, directed graph by setting $x_{ij} = 0$ whenever (v_i, v_j) is not an edge in the graph. For the case of a non-directed graph, we let $x_{ij} = x_{ji}$ whenever (v_i, v_j) is an edge. A non-weighted graph is the case where all nonzero weights are 1.

Question: What is the rank of L for a directed graph with positive real weights (a weight of 0 would indicate the absence of an edge)? If there is a spanning tree directed into some vertex, the theorem shows that the rank is then one less than the number of vertices of G. It also easily follows from the theorem that the rank is greater than or equal to one less than the number of vertices in any spanning tree directed into some vertex.