

Prop. Let (Γ, s) be a sandpile graph. Then

$$\begin{aligned} \varphi: Cl(\Gamma) &\longrightarrow \mathbb{Z} \oplus \text{critical}(\Gamma) \\ D &\longmapsto (\deg D, D - (\deg D)s) \end{aligned}$$

is an isomorphism.

PF/ We have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \frac{\mathbb{Z}V}{\ker \Delta} & \xrightarrow{\Delta} & \mathbb{Z}V & \rightarrow & Cl(\Gamma) \rightarrow 0 \\ & & \uparrow id & & \uparrow z & & \uparrow \\ 0 & \rightarrow & \frac{\mathbb{Z}V}{\ker \Delta} & \xrightarrow{\Delta_0} & \mathbb{Z}V_0 & \rightarrow & \text{critical}(\Gamma) \rightarrow 0 \end{array}$$

where id is the identity and $z(v) = v$. Since $\ker z = \{0\}$ and

$\text{coh}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$, the snake lemma gives a short exact sequence

(2)

$$D \mapsto \deg(D)$$

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{critical}(\Gamma) & \rightarrow & Cl(\Gamma) & \xrightarrow{\deg} & \mathbb{Z} \rightarrow 0 \\ [D] & \mapsto & [D] & & & & \\ & & [D] & \mapsto & \deg D & & . \end{array}$$

However, this sequence splits: Define $Cl(\Gamma) \rightarrow \text{critical}(\Gamma)$

$$[D] \mapsto D - (\deg D)s.$$

This mapping is well-defined since

$$D \sim E \iff D - E = \Delta\sigma \text{ for some } \sigma \iff (D - (\deg D)s) - (E - \deg(E)s) = \Delta\sigma$$

for some σ .

The result follows.



(3)

Recall that earlier, we showed there is an exact sequence

$$0 \rightarrow \frac{\mathbb{Z}}{\tau_s \mathbb{Z}} \rightarrow S(\Gamma, s) \rightarrow \text{critical}(\Gamma) \rightarrow 0$$

$$c \mapsto c - (\deg c)s$$

where $\tau_v = \# \text{ directed spanning trees into } v$ and $\tilde{\tau}_v = \tau_v / \gcd_v(\tau_v)$.

Therefore, if $\tilde{\tau}_s = 1$ (as in the case of an Eulerian graph),

we have $\Psi: S(\Gamma, s) \rightarrow \text{Cl}_d(\Gamma)$ is an isomorphism.

$$c \mapsto c + (d - \deg(c))s$$

Even if $\tilde{\tau}_s \neq 1$, we still get that Ψ is surjective.