# Math 411



From last time:

Cycles: $\sum n_i V_i \in Z_r X$

rational equivalence: $V \sim W$

$A^r X := Z_{n-r} X / \sim$

Chow ring: $A^* X = \bigoplus A^l X$ product: $[V] \cdot [W] = [V \cap W] \in A^l X$.

flag: $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_r$ linear subspaces of $\mathbb{P}^n$

Schubert variety: $\mathfrak{S}(A_0, \ldots, A_r) = \{ L \in \mathbb{G}_r \mathbb{P}^n : \dim(L \cap A_i) \geq i \}$

Schubert class: $(a_0, \ldots, a_r) = [\mathfrak{S}(A_0, \ldots, A_r)] \in A^* \mathbb{G}_r \mathbb{P}^n$

- depends only on $\dim A_i$
- $\operatorname{codim}(a_0, \ldots, a_r) = (r+1)(n-r) - \sum (a_i - i)$
- $A^* \mathbb{G}_r \mathbb{P}^n$ is free Abelian on $\{(a_0, \ldots, a_r) : 0 \leq a_0 < \cdots < a_r \leq n\}$

(Note: If $k = \mathbb{C}$, then $A^l \mathbb{G}_r \mathbb{P}^n \cong H^{2l} \mathbb{G}_r \mathbb{P}^n$.)

When $\dim(L \cap A_i) \geq i$ imposes no condition.

Recall that for linear subspaces $L, M$ of $\mathbb{P}^n$, $\dim L \cap M \geq \dim L + \dim M - n$.

So if $L \in \mathbb{G}_r \mathbb{P}^n$, we have

$$\dim(L \cap A_i) \geq r + a_i - n.$$

Hence, $\dim(L \cap A_i) \geq i$ for all $L \in \mathbb{G}_r \mathbb{P}^n$ if

$$r + a_i - n \geq i,$$

i.e., if

$$a_i \geq n - r + i.$$

Example $\mathbb{G}_3 \mathbb{P}^6$

$(3,4,5) = [\mathbb{G}_3 \mathbb{P}^6]$

$(2,4,\underline{5})$ $(1,3,\underline{5})$

no condition

no condition

Example $G_1\mathbb{P}^3$ $\dim = 4$ $0 \le a_0 < a_1 \le 3$

| codimension | class | condition |
|---|---|---|
| 0 | $(2,3) = \{0,0\}$ | no condition |
| 1 | $(1,3) = \{1,0\}$ | meet a line |
| 2 | $(0,3) = \{2,0\}$ | pass through a point |
| 2 | $(1,2) = \{1,1\}$ | lie in a plane |
| 3 | $(0,2) = \{2,1\}$ | pass through a point an lie in a plane |
| 4 | $(0,1) = \{2,2\}$ | be a certain line |

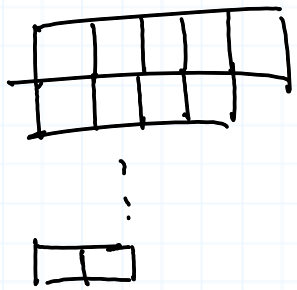
Partition associated with $(a_0,\dots,a_r)$: Let $\lambda_i = n-r-(a_i-i)$ $\forall i$.

Write $\{\lambda_0,\dots,\lambda_r\} = (a_0,\dots,a_r) \in A^* G_r\mathbb{P}^n$.

- $\operatorname{codim}\{\lambda_0,\dots,\lambda_r\} = |\lambda| := \sum \lambda_i$
- $n-r \ge \lambda_0 \ge \cdots \ge \lambda_r = 0$.

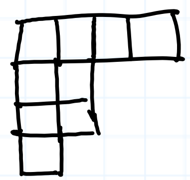
# Multiplication in $A^* G_r \mathbb{P}^n$

For each $\lambda : \lambda_0 \geq \cdots \geq \lambda_r$ there is an associated Young diagram



$\lambda_0$ boxes
$\lambda_1$ boxes
$\vdots$
$\lambda_r$ boxes

Example $\{4,2,2,1\}$



Thm. For $\{\lambda\}, \{\mu\} \in A^* G_r \mathbb{P}^n$

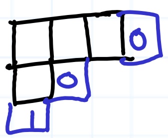
$$\{\lambda\} \cdot \{\mu\} = \sum_{\{\nu\} \in A^* G_r \mathbb{P}^n} N_{\lambda\mu\nu} \{\nu\}$$

where $N_{\lambda\mu\nu}$ is the Littlewood-Richardson number, i.e. the number of strict $\mu$ expansions of $\lambda$ giving $\nu$.

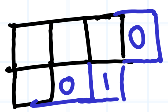
**Def.** $\mu$-expansion of $\lambda$: Start with the Young diagram for $\lambda$. Build the $\mu$-expansion as follows: at the $i^{\text{th}}$ step, 1 add $\mu_i$ boxes to existing rows or the the bottom; 2 no two in the same column, 3 write the number $i$ in each box. The result must be a valid Young diagram.

The expansion is strict if reading off the numbers right-to-left, top-to-bottom, at each step, each integer $k$ occurs at least as many times as $k+1$.

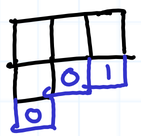
**Example** Some $\{2,1\}$ expansions of $\{3,1\}$:



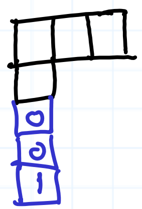
strict



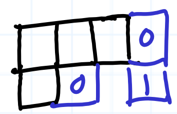
strict



not strict



not allowed
(two 0s in same column)



not allowed
(not a Young diagram)

# Multiplication table for $G_1 \mathbb{P}^3$

| codim | $*$ | $1$ | $\ydiagram{1}$ | $\ydiagram{2}$ | $\ydiagram{1,1}$ | $\ydiagram{2,1}$ | $\ydiagram{2,2}$ |
|---|---|---|---|---|---|---|---|
| 0 | $1$ | $1$ | $\ydiagram{1}$ | $\ydiagram{2}$ | $\ydiagram{1,1}$ | $\ydiagram{2,1}$ | $\ydiagram{2,2}$ |
| 1 | $\ydiagram{1}$ | $\ydiagram{1}$ | $\ydiagram{2} + \ydiagram{1,1}$ | $\ydiagram{2,1}$ | $\ydiagram{2,1}$ | $\ydiagram{2,2}$ | $0$ |
| 2 | $\ydiagram{2}$ | $\ydiagram{2}$ | $\ydiagram{2,1}$ | $\ydiagram{2,2}$ | $0$ | $0$ | $0$ |
| 2 | $\ydiagram{1,1}$ | $\ydiagram{1,1}$ | $\ydiagram{2,1}$ | $0$ | $\ydiagram{2,2}$ | $0$ | $0$ |
| 3 | $\ydiagram{2,1}$ | $\ydiagram{2,1}$ | $\ydiagram{2,2}$ | $0$ | $0$ | $0$ | $0$ |
| 4 | $\ydiagram{2,2}$ | $\ydiagram{2,2}$ | $0$ | $0$ | $0$ | $0$ | $0$ |

Example:
$$\left(\ydiagram{1}\right)^4 = \ydiagram{1}^2\left(\ydiagram{2} + \ydiagram{1,1}\right) = \ydiagram{1}\left(2\,\ydiagram{2,1}\right) = \underline{\underline{2}}\,\ydiagram{2,2}$$

Interpretation: $\ydiagram{1}$ = meet a line

There are 2 lines in $\mathbb{P}^3$ meeting 4 given (generic) lines.