Math 411 Toric Varieties

lattice N = free $\mathbb{Z}$-module of rank n; so $N \cong \mathbb{Z}^n$ as $\mathbb{Z}$-modules

$$N \subset N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$$

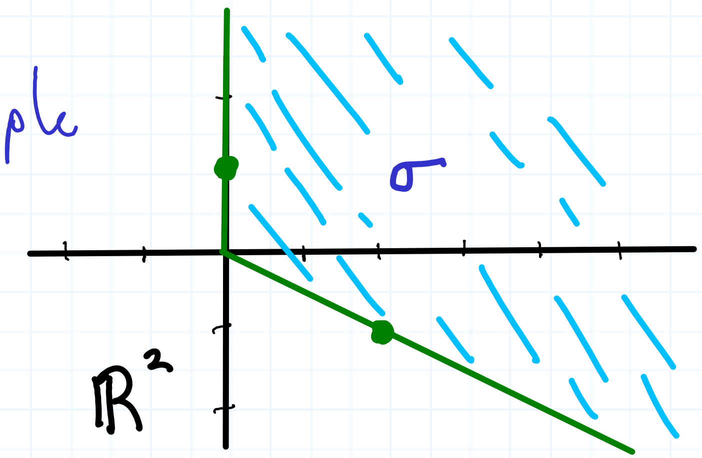
cone σ = strongly convex, rational polyhedral cone in $N_{\mathbb{R}}$

does not contain a line through the origin

generated by a finite number of lattice points

closed under $+$ and multiplication by $\lambda \in \mathbb{R}_{\geq 0}$

Example



$$\sigma = \mathbb{R}_{\geq 0}(0,1) + \mathbb{R}_{\geq 0}(2,-1)$$

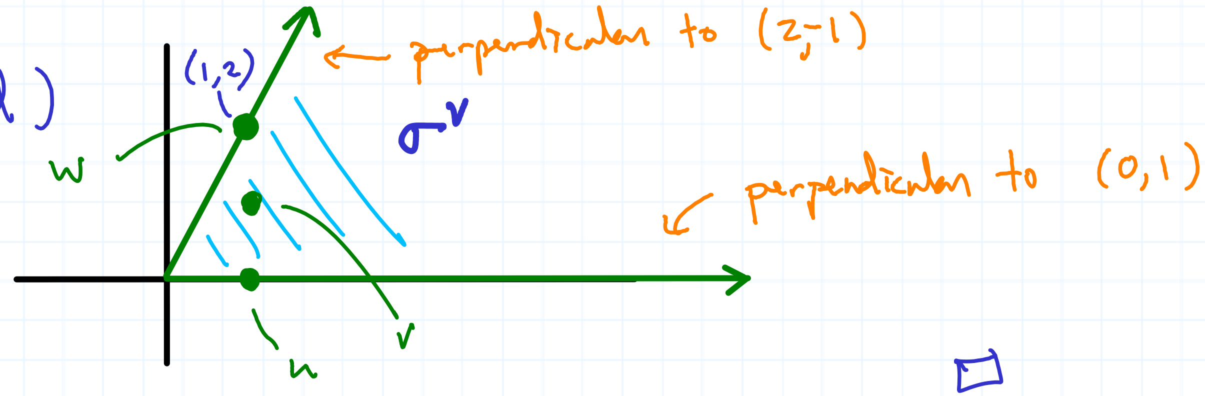
□

dual lattice $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ Choosing $\mathbb{Z}$-basis e_1, e_2 for N, we get the dual basis e_1^*, e_2^* for M

$$\begin{array}{ccc} \overset{\mathbb{Z}^n \cong}{M} \times \overset{\cong \mathbb{Z}^n}{N} \longrightarrow \mathbb{Z} & & \overset{\mathbb{R}^n \cong}{M_{\mathbb{R}}} \times \overset{\cong \mathbb{R}^n}{N_{\mathbb{R}}} \longrightarrow \mathbb{R} \\ (u,v) \longmapsto u(v) & \rightsquigarrow & (u,v) \longmapsto u(v) \end{array}$$

dual cone $\sigma^\vee = \{ u \in M_{\mathbb{R}} : (u,v) \geq 0 \ \forall v \in \sigma \}$

Example (continued)



$\square$

Semigroup associated with σ; $S_\sigma := \sigma^\vee \cap M$

Example (continued) S_σ is generated (additively, i.e., over $\mathbb{Z}_{\geq 0}$) by $u = (1,0)$, $v = (1,1)$, $w = (1,2)$ and there is one relation: $u + w = 2v$.

$\square$

Semigroup algebra $\mathbb{C}[S_\sigma] := \mathbb{C}[e_u : u \in \sigma^\vee]$

$$e_u \cdot e_v := e_{u+v}$$

Example (continued)

$u = (1,0)$, $x = e_u$

$v = (1,1)$, $y = e_v$

$w = (2,1)$, $z = e_w$

$$\mathbb{C}[S_\sigma] \cong \mathbb{C}[x,y,z]/(xz - y^2)$$

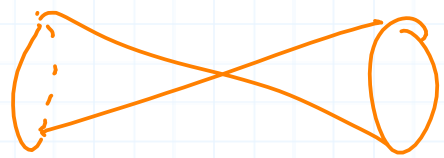
$\square$

Toric variety If $\mathbb{C}[S_\sigma] \cong \mathbb{C}[x_1,\dots,x_n]/I$ for some ideal $I \subseteq \mathbb{C}[x_1,\dots,x_n]$,

then $U_\sigma := Z(I) := \{ p \in \mathbb{C}^n : f(p) = 0 \ \forall f \in I \}$.

Example (continued)

$$U_\sigma := \{ (x,y,z) \in \mathbb{C}^3 : xz = y^2 \}$$



$U_\sigma \cap \mathbb{R}^3$

$\square$

Note: $\mathbb{C}[S_\sigma]$ can be interpreted as a space of functions on U_σ.

If $p \in U_\sigma$, $f \in \mathbb{C}[x_1,\dots,x_n]/I \cong \mathbb{C}[S_\sigma]$, and $g \in I$,

then $(f+g)(p) := f(p) + g(p) = f(p) + 0 = f(p)$.

Fan Δ = collection of non-overlapping cones, closed under intersection.

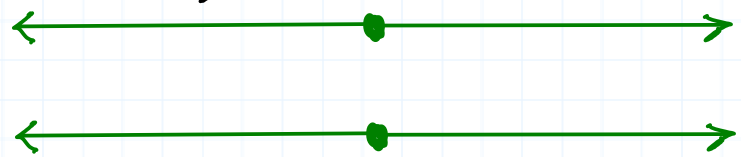
Example $\underline{\underline{\mathbb{P}^1}}$

$\sigma_2 = \mathbb{R}_{\geq 0}(-1) = \mathbb{R}_{\leq 0}$ $\qquad$ $\sigma_1 = \mathbb{R}_{\geq 0}(1) = \mathbb{R}_{\geq 0}$

$\tau = \sigma_1 \cap \sigma_2 = \{0\}$

$N = \mathbb{Z}$

$\sigma_2^\vee$ $\qquad$ $\sigma_1^\vee$



$\longleftarrow \tau^\vee \longrightarrow$

$M = \mathbb{Z}$

$x := e_{(1,0)}$

$y := e_{(0,1)}$

$x \mapsto x$

$y \mapsto \frac{1}{x}$

$$\begin{array}{l} \mathbb{C}[S_{\sigma_1}] \cong \mathbb{C}[x] \searrow \\ \qquad\qquad\qquad\qquad \mathbb{C}[x,y]/(xy-1) \cong \mathbb{C}[x, \tfrac{1}{x}] = \mathbb{C}[S_\tau] \\ \mathbb{C}[S_{\sigma_2}] = \mathbb{C}[y] \nearrow \end{array}$$

$$\Big\downarrow \ \mathrm{Hom}_{\mathbb{C}}(\bullet, \mathbb{C})$$

$$\begin{array}{ccccc} & x & & & \\ & & \longleftrightarrow \ x & & \\ U_{\sigma_1} \cong \mathbb{C} & \longleftarrow & & \mathbb{C}^* \cong \{(x,y) \in \mathbb{C}^2 : xy = 1\} = U_\tau & \\ & & & x \longleftrightarrow (x,y) & \\ U_{\sigma_2} \cong \mathbb{C} & \longleftarrow & & x \longmapsto (x, \tfrac{1}{x}) & \\ \tfrac{1}{x} & & & & \end{array}$$

Toric variety $X(\Delta)$: 2 copies of $\mathbb{C}$ glued together by $x \longmapsto \frac{1}{x}$. The overlap is $\mathbb{C}^*$.