Def. A semi-Riemannian manifold is a manifold $M$ with a scalar product $\langle , \rangle_p$ on $T_p M \forall p \in M$, varying smoothly with $p$. If the scalar product has index 0, then $M$ is Riemannian.

We saw that a scalar product on a vector space $V$ corresponds with an element of the symmetric product $S^2 V^*$. Thus, we can define a semi-Riemannian manifold as a manifold with a section of the vector bundle $S^2 T^* M$ such that the corresponding bilinear form on $T_p M$ is nondegenerate $\forall p$. Consider the entries in the matrix defining $\langle , \rangle$ locally.
Let $M$ be an oriented semi-Riemannian $n$-manifold. For each $p \in M$, we get a volume form $\omega_p \in \Omega^n M$ and a $\ast$-operator $\ast : \Lambda^k T^*_p M \to \Lambda^{n-k} T^*_p M$ varying smoothly with $p$. These glue together to give a volume form $\omega \in \Omega^n M$ and a $\ast$-operator $\ast : \Omega^k M \to \Omega^{n-k} M$.

We get an induced coderivative mapping:

\[
\begin{align*}
\Omega^k M & \xrightarrow{\delta} \Omega^{k+1} M \\
\ast & \downarrow \quad \downarrow \ast \\
\Omega^{n-k} M & \xrightarrow{(-1)^k \delta} \Omega^{n-k-1} M
\end{align*}
\]

**Lemma** $\ast \ast = (-1)^{n(n-k)+1} \text{Id}$. Hence, $\ast$ is an isomorphism.

Pf/ HW. $\square$
Def. The codifferential $\delta_k : \Omega^k M \rightarrow \Omega^{k-1} M$ is defined by

$$\delta_k = (-1)^k \circ d \circ \ast^{-1}$$

Goal. We would like to use $\ast$ to get an isomorphism $H^k M \rightarrow H^{n-k} M$ for closed Riemannian manifolds.

Recall $H^k M = \ker (\delta^k : \Omega^k M \rightarrow \Omega^{k+1} M) / \text{im}(\delta^{k-1} M \rightarrow \delta^k M)$.

Let $\eta \in \Omega^k M$ with $d \eta = 0$, i.e., $\eta$ is a cocycle.

Q. Is $\ast \eta$ a cocycle?

There is kind of commutation of $\ast$ and $d$:

- $\delta \ast \eta = \pm \ast \delta \ast \eta = \pm \ast \delta \ast \eta = \pm \ast \delta \ast \eta$

There is kind of commutation of $\ast$ and $d$:

- $d \ast \eta = \pm \ast \delta \ast \eta = \pm \ast \delta \ast \eta = \pm \ast \delta \ast \eta$

Therefore $\eta = \pm \ast \delta \ast \eta$.
So, \( d \ast \eta = \pm \delta \eta \) and, similarly, \( \delta \ast \eta = \pm \ast d \eta \).

Thus, we are led to consider:

**Def.** The **harmonic** \( k \)-**forms** on \( M \) are

\[
\mathcal{H}ar(M) = \{ \eta \in \Omega^k M : d \eta = 0 \text{ and } \delta \eta = 0 \}.
\]

**Thm.** If \( M \) is a closed Riemannian manifold

\[
\mathcal{H}ar^k M \rightarrow H^k M
\]

\[
\eta \mapsto [\eta]
\]

is an isomorphism.

**Pf/ Punt.** \( \Box \)
Prop. \( \ast : \operatorname{Har}^k M \to \operatorname{Har}^{n-k} M \) exists and is an isomorphism.

Pf/ \( \ast H \in \operatorname{Har}^k M \). So \( dH = 0 \) and \( \delta H = 0 \). Then \( d \ast H = \pm \delta dH = 0 \) and \( \delta \ast H = \pm \ast dH = 0 \).

Hence, \( \ast H \) is harmonic. Hence, \( \ast : \operatorname{Har}^k M \to \operatorname{Har}^{n-k} M \) makes sense. It is an isomorphism since \( \ast \ast = \pm \operatorname{id}. \square \)

Thm. (Poincaré duality) If \( M \) is a closed Riemannian manifold, then
\[
\ast : H^k M \to H^{n-k} M.
\]

Example: \( H^0 S^n = \mathbb{R} \Rightarrow H^n S^n = \mathbb{R} \) (spanned by the volume form).

\textbf{Note:} Every manifold can be given a Riemannian metric by using the charts and a partition of unity. So Poincaré duality holds for oriented closed manifolds.
Cor. If \( M \) is a closed, connected, oriented \( n \)-manifold, then

\[
\int_{\partial M} \omega = \int M
\]

is an isomorphism.

**Pf/** \( H^n M = H^0 M = \mathbb{R} \Rightarrow H^n M = \mathbb{R} [w_M] \), where \( w_M \) is the volume form. Note that locally \( \int_M w_M = \int dx_1 \wedge \ldots \wedge dx_n = \int_A 1 = \text{vol}(A) \). So \( \int_M w_M \) really does give the volume. In particular, \( \int_M w_M \neq 0 \). \( \square \)