Continuing from last time:

\[ 0 \rightarrow \mathcal{A}^k W \rightarrow \mathcal{A}^k U \times \mathcal{A}^k V \rightarrow \mathcal{A}^k (U \cap V) \rightarrow 0 \]

\[ \omega \mapsto (i^*_u \omega, i^*_v \omega) \]

\[ (\xi, \eta) \mapsto j^*_u \xi - j^*_v \eta \]

Thm. (※) is a short exact sequence.

Pf/ At \( \mathcal{A}^k W \) If the restriction of the form \( \omega \in \mathcal{A}^k W \) to \( U \) and to \( V \) is zero, then \( \omega = 0 \). This is obvious by taking local coordinates. Hence, \( \mathcal{A}^k W \rightarrow \mathcal{A}^k U \times \mathcal{A}^k V \) is injective.

At \( \mathcal{A}^k U \times \mathcal{A}^k V \) Let \( (\xi, \eta) \in \mathcal{A}^k U \times \mathcal{A}^k V \) and suppose \( j^*_u \xi - j^*_v \eta = 0 \).

Define \( \omega \in \mathcal{A}^k W \) by \( \omega(p) = \begin{cases} \xi(p) & \text{if } p \in U \\ \eta(p) & \text{if } p \in V \end{cases} \). Then \( \omega \mapsto (\xi, \eta) \)
under restriction. Thus, \( \ker (\mathcal{L}^k(U \times V) \to \mathcal{L}^k(UNV)) \leq \text{im} (\mathcal{L}^kM \to \mathcal{L}^kU + \mathcal{L}^kV) \).

For the opposite inclusion, given any \( \omega \in \mathcal{L}^kM \), we have
\[
\omega \mapsto (z^u \omega, z^v \omega) \mapsto j^* z^u \omega - j^* z^v \omega = (z^uj^* \omega - (z^uj^*)^* \omega = 0 \quad \text{since}
\]
\[
z^uj^* (x) = z^v j^* (x) \quad \forall x \in UNV.
\]

At \( \mathcal{L}^kUNV \) let \( \omega \in \mathcal{L}^k(UNV) \). Take a partition of unity subordinate to the covering \( \{ U, V \} \) of \( M \). That is take functions,
\[
\lambda_u, \lambda_v : M \to \mathbb{R} \quad \text{with} \quad \text{supp} \lambda_u \subset U, \text{supp} \lambda_v \subset V \quad \text{and}
\]
\[
\lambda_u(m) + \lambda_v(m) = 1 \quad \text{for all} \quad m \in M. \quad \text{Define} \quad w_u = \lambda_u \omega \in \mathcal{L}^kU
\]
and \( w_v = \lambda_v \omega \in \mathcal{L}^kV \). Thus, for instance, \( w_u(p) = 0 \) for \( p \in U \cap V \).

Then \( (w_u, -w_v) \mapsto \omega_u - (-w_v) = w_u + w_v = \lambda_u \omega + \lambda_v \omega = (\lambda_u \lambda_v) \omega = \omega \).
The short exact sequence of chain complexes in the theorem induces a long exact sequence of cohomology groups.

**Vector Fields on Spheres**

**Def.** A vector field on a manifold $M$ is a smooth section of the tangent bundle:

$$\begin{align*}
v : M &\to TM \\
p &\mapsto v_p \in T_p M
\end{align*}$$

Locally, $v(p) = \sum a_i(p) \frac{\partial}{\partial x_i}|_p$ when $a_i$ is a smooth $\mathbb{R}$-valued function.
Thm. (Hairy ball thm.) Every vector field on an even-dimensional sphere has at least one zero.

PF/ Suppose $v$ is a non-vanishing v.f. on $S^n = \{ x \in \mathbb{R}^n : \sum x_i^2 = 1 \}$, $n$ even. Then the antipodal mapping $\tau : S^n \to S^n$, $\tau(p) = -p$, is homotopic to the identity mapping:

$$h(t,x) = x \cos(t\pi) + \frac{v(x)}{|v(x)|} \sin(t\pi), \quad t \in [0,1].$$

Here we are identifying $T_x S^n$ with vectors perpendicular to $x$ in $\mathbb{R}^{n+1}$ (thinking of tangent vectors in terms of curves through $x$). Thus, for fixed $x$, the image of $h(t,x)$ is a circle on the plane spanned by the perpendicular vectors $x$ and $v(x)$. 

$h(0,x) = x$

$h(1,x) = -x$
By the homotopy invariance theorem, if \( w \in \Omega^n S^n \) (automatically a cocycle since \( \dim S^n = n \)), then \( \tau^* = \text{id}^* : H^n S^n \rightarrow H^n S \) implies \( \tau^* w = \text{id}^* w = w \mod d(\Omega^n S^n) \), i.e. \( \tau^* w - w = d\eta \) for some \( \eta \in \Omega^{n-1} S^n \). Therefore,

\[
\int_{S^n} \tau^* w - \int_{S^n} w = \int_{S^n} d\eta = \int_{S^{n+1}} \eta = \int_{S^n} \eta = 0,
\]
i.e.,

\[
\int_{S^n} \tau^* w = \int_{S^n} w.
\]

However, if \( n \) is even, \( \tau^* \) is an orientation reversing diffeomorphism, so

\[
\int_{S^n} \tau^* w = -\int_{S^n} w.
\]

So on an even-dimensional sphere, we have shown that \( \int_{S^n} w = 0 \)
for all \( w \in S^n \). However, this is not true—consider a bump function. Therefore, \( \nu \) cannot exist.

\( \star \)

1. \( I^n \) is orientation reversing

Let \( D^{n+1} \) denote the \((n+1)\)-dimensional unit ball in \( \mathbb{R}^{n+1} \). The orientation on \( D^{n+1} \) is induced by the positive orientation on \( \mathbb{R}^{n+1} \). The orientation on \( S^n = \partial D^{n+1} \) is the orientation induced boundary orientation: \( \nu, \ldots, \nu_n \in T_p S^n \) is positive if \( v, v_1, \ldots, v_n \) is positive for any outward-pointing tangent vector \( v \).

The antipodal map \( \tau(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, -x_{n+1}) \) has Jacobian with determinant \(-1\) if \( n \) is even, thus, is orientation reversing on \( \mathbb{R}^{n+1} \). Therefore, it is orientation reversing on \( D^{n+1} \) and \( \partial D^{n+1} = S^n \).
2. If \( f: M \to N \) is an orientation-reversing diffeomorphism between \( n \)-manifolds and \( w \in \mathbb{R}^n \), then \( \int_M f^* w = -\int_N w \).

This is a local question, so we may assume that \( M \) and \( N \) open subsets of \( \mathbb{R}^n \) with the usual positive orientation and \( w = dx_1 \wedge \cdots \wedge dx_n \). Then

\[
\int_M f^* w = \int_M a \circ f \det df_1 \wedge \cdots \wedge df_n = \int_M a \circ f \det(Jf) \, dx_1 \wedge \cdots \wedge dx_n
\]

\[
= \int_M a \circ f \, \det(Jf) = -\int_N a \circ f |\det Jf| \uparrow \text{change of variables formula} \downarrow \det Jf < 0 \text{ since } f \text{ is orientation-reversing}
\]

\[
= -\int_N a = -\int_N w. \quad \blacksquare
\]
What about the number of zeros of a vector field?

Let $M$ be a compact Riemann surface (a multi-holed donut).

**Quasi-Def.** Indices of zeros of vector fields on a surface $M$ change in angle the vector field makes as one travels counter-clockwise along a simple closed curve containing the zero and only that zero$/2\pi$.

**Example**

-1

+1

+1

Thm. Consider a vector field on $M$ with a finite number of zeros. Then the sum of the indices of a vector field on $M$ is $2 - 2g$ where $g$ is the genus (number of holes) of $M$.

Note: $H^1(M \cong \mathbb{R}^{2g}$ (use Mayer-Vietoris)