Def. A chain complex is a sequence of mappings of vector spaces

$$\cdots \xrightarrow{d} C_i \xrightarrow{d} C_{i+1} \xrightarrow{d} C_{i+2} \xrightarrow{d} \cdots$$

such that $d^2 = 0$. We would denote this chain complex by $C^\cdot$. A mapping of chain complexes $f: B^\cdot \to C^\cdot$ is a collection of mappings $f_i : B_i \to C_i$ such that $d f_i = f_{i+1} d$:

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & B_i \arrow[r, "d"] \arrow[d, "f_i"] & B_{i+1} \arrow[r, "d"] \arrow[d, "f_{i+1}"] & B_{i+2} \arrow[r, "d"] \arrow[d, "f_{i+2}"] & \cdots \\
\cdots \arrow[r, "d"] & C_i \arrow[r, "d"] & C_{i+1} \arrow[r, "d"] & C_{i+2} \arrow[r, "d"] & \cdots
\end{tikzcd}$$

Even though their names are the same, the d's are distinct mappings; there domains are usually distinct, for example.

The k^{th} cohomology group for a chain $C^\cdot$ is

$$H^k C^\cdot = \frac{\ker(C_k \xrightarrow{d} C_{k+1})}{\operatorname{im}(C_{k-1} \to C_k)}$$

A sequence of mappings of vector spaces $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B if $\operatorname{im} f = \ker g$. Thus, the k^{th} cohomology group of a chain complex $C^\cdot$ measure the exactness at C_k.

A short exact sequence (s.e.s.) of vector spaces is a sequence of mappings of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

exact at A, B, and C. This is the same as saying (1) f is injective, g is surjective, and $\operatorname{im} f = \ker g$, so that $B/f(A) \approx C$.

A short exact sequence of chain complexes is a sequence of mappings of complexes

$$0 \to A^{\cdot} \to B^{\cdot} \to C^{\cdot} \to 0$$

such that $0 \to A_i \to B_i \to C_i \to 0$ is a s.e.s. for all i.

Prop. A s.e.s. of chain complexes $0 \to A^{\cdot} \to B^{\cdot} \to C^{\cdot} \to 0$ induces a (long) exact sequence of homology groups:

$$\cdots \to H^{k-1}(C) \xrightarrow{\delta} H^k(A) \to H^k(B) \to H^k(C) \xrightarrow{\delta} H^{k+1}(A) \to \cdots$$

Description of δ:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & A_{k+1} \arrow[u] \arrow[r, "f"] & B_{k+1} \arrow[u] \arrow[r, "g"] & C_{k+1} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & A_k \arrow[u, "d"] \arrow[r, "f"] & B_k \arrow[u, "d"] \arrow[r, "g"] & C_k \arrow[u, "d"] \arrow[r] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Let $c \in C_k$ with $dc = 0$ (representing $[c] \in H^k C^\cdot$).
Since g is surjective, $\exists\, b \in B_k$ such that $gb = c$. By commutativity, $g\,db = dg\,b = dc = 0$. By exactness, $\exists\, a \in A_{k+1}$ s.t. $f(a) = db$. By commutativity, $fda = dfa = d^2 b = 0$.
Since f is injective, $da = 0$. So $a \in \ker(A_{k+1} \xrightarrow{d} A_{k+2})$.
Define $\delta^* [w] = [u]$. $\square$

Pf of Prop./ Exercise. $\square$

Let M be a manifold. Let $M = U \cup V$ where U and V are open subsets of M. Let $i_U : U \to M$, $i_V : V \to M$, $j_U : U \cap V \to U$, and $j_V : U \cap V \to V$ be the inclusion mappings.
Let $U \amalg V$ denote the disjoint union of U and V, the coproduct in the category of sets. Then

are inclusion mappings $M \xleftarrow{i_U \oplus i_V} U \amalg V \underset{j_V}{\overset{j_U}{\Leftarrow}} U \cap V$. Applying the contravariant functor $\Omega^k(\cdot)$, we get restriction mappings

$$\Omega^k M \xrightarrow{i_U^* \times i_V^*} \Omega^k U \times \Omega^k V \underset{j_V^*}{\overset{j_U^*}{\rightrightarrows}} \Omega^k(U \cap V)$$

(restriction of forms = pullback along inclusions). Consider the sequence

$$(\star) \qquad \begin{array}{ccccccccc} 0 \to & \Omega^k M & \longrightarrow & \Omega^k U \times \Omega^k V & \longrightarrow & \Omega^k(U\cap V) & \to 0 \\ & \omega & \longmapsto & (i_U^* \omega, i_V^* \omega) & & & \\ & & & (\xi, \eta) & \longmapsto & j_U^* \xi - j_V^* \eta & \end{array}$$

Thm. $(\star)$ is a short exact sequence.

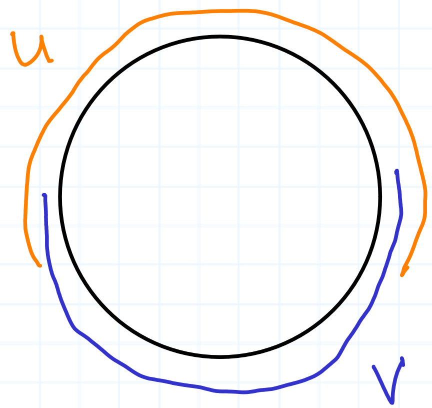
Pf/ Next time. $\square$

☆ Mayer - Vietoris ☆

$$\cdots \to H^{k-1}(U\cap V) \xrightarrow{\delta} H^k(M) \to H^k U \times H^k V \to H^k(U\cap V) \xrightarrow{\delta} H^{k+1}(M) \to \cdots$$

Application: Computation of $H^1 S^1$.

Let S^1 be the unit circle. Cover S^1 by open sets U, V as shown:



Mayer-Vietoris gives the exact sequence:

$$0 \to H^0(S^1) \to H^0(U)\times H^0(V) \to H^0(U\cap V) \xrightarrow{\delta} H^1(S^1) \to H^1(U)\times H^1(V) \to H^1(U\cap V) \to 0$$

Now H^0 gives the number of connected components and $H^1 = 0$ for a contractible manifold. Hence, the sequence becomes

$$0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}^2 \to H^1(S^1) \to 0$$

Exactness then forces $H^1(S^1) = \mathbb{R}$.

Similarly, one may show for the n-sphere:

$$H^k S^n = \begin{cases} \mathbb{R} & \text{if } k = 0 \text{ or } n \\ 0 & \text{otherwise.} \end{cases}$$