Properties of the integral

M oriented n-manifold, $\omega \in \Omega^n M$

* If ω is locally integrable, i.e. on an open neighborhood about each point, then ω is integrable if ω has compact support, e.g. if M is compact.

  support ← $p \in M$ s.t. $\omega_p \neq 0$

* Let $-M$ be M with the opposite orientation. Then $\int_{-M} \omega = -\int_M \omega$.

  $M \quad (U,h) \qquad \omega|_U = a\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$

  $-M \quad (U,\tilde{h}) \qquad \omega|_U = b\, dx_2 \wedge dx_1 \wedge \cdots \wedge dx_n$, say $\Rightarrow b = -a$

* If $f: M \to N$ is an orientation preserving diffeomorphism and $\omega \in \Omega^n N$ is integrable, then $f^*\omega$ is integrable on M and

$$\int_M f^*\omega = \int_N \omega$$

Pf/ Take a chart (U,h) at $f(p)$ in N. Then $(f^{-1}(U), h\circ f)$ is a chart at p.

$$\begin{tikzcd}
 & M & N & \\
 & \cup\!\!\mid & \cup\!\!\mid & \\
(f^*\omega)_p & f^{-1}(U) \arrow[r, "f"] \arrow[d, "h\circ f"'] & U \arrow[d, "h"] & \omega_{f(p)} \\
 & \mathbb{R}^n \arrow[r, "id"] & \mathbb{R}^n & \\
 & x_1, \dots, x_n & y_1, \dots, y_n &
\end{tikzcd}$$

Coefficient of $dx_{1,p} \wedge \cdots \wedge dx_{n,p}$ in the local expression for $(f^*\omega)_p$:

see HW

$$(f^*\omega)_p\left(\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p\right) \overset{*}{=} \omega_{f(p)}\left(f_*\left(\frac{\partial}{\partial x_1}\right)_p, \dots, f_*\left(\frac{\partial}{\partial x_n}\right)_p\right)$$

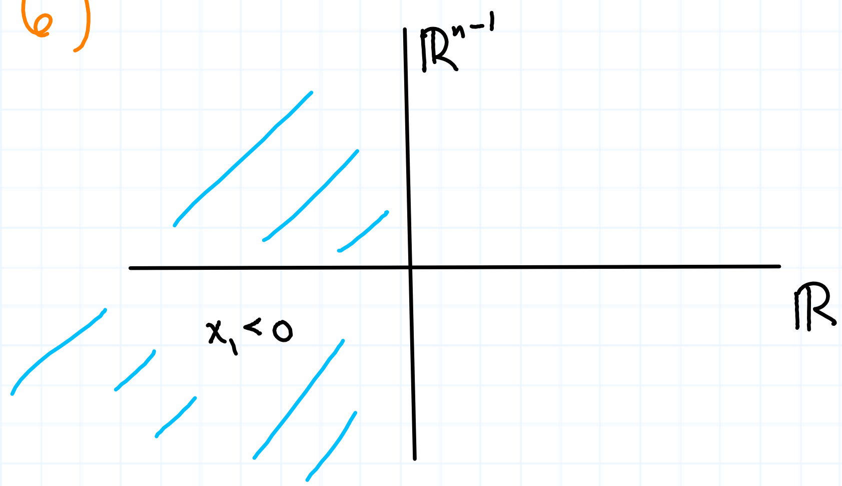
$$= \omega_{f(p)}\left(\left(\frac{\partial}{\partial y_1}\right)_{f(p)}, \dots, \left(\frac{\partial}{\partial y_n}\right)_{f(p)}\right),$$

which is the coefficient of $dy_{1,f(p)} \wedge \cdots \wedge dy_{n,f(p)}$ in the local expression of $\omega_{f(p)}$. Hence, the corresponding local integrals will be equal.

Next goal: Stokes' thm. $\int_{\partial M} \omega = \int_M d\omega$.

Manifolds with boundary (Chapter 6)

Def. $\mathbb{R}^n_- := \{ (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \leq 0 \}$

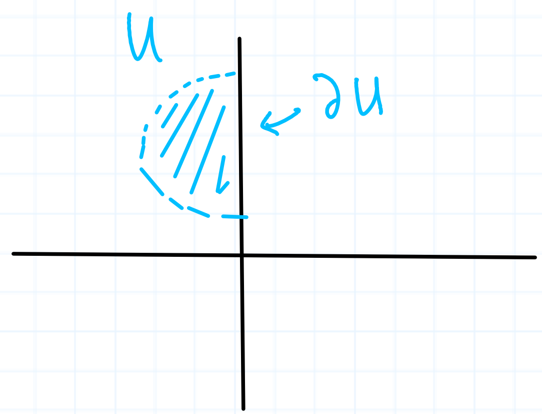


with the subspace topology inherited from $\mathbb{R}^n$, i.e. the open subsets of $\mathbb{R}^n_-$ are exactly sets of the form

$W \cap \mathbb{R}^n_-$ where W is an open subset of $\mathbb{R}^n$.

Def. If $U \subseteq \mathbb{R}^n_-$, then the boundary of U is

$$\partial U = U \cap \{x_1 = 0\} = U \cap [\{0\} \times \mathbb{R}^{n-1}]$$



Def. (differentiability) Let $U \subseteq \mathbb{R}^n_-$ be open. A function $f: U \to \mathbb{R}^m$ is **differentiable** at $p \in U$ if $\exists$ open $W \subseteq \mathbb{R}^n$ with $p \in W$ and differentiable $g: W \to \mathbb{R}^m$ (in the usual sense) such that $f|_{U \cap W} = g|_{U \cap W}$.

Def. $U, V \overset{\text{open}}{\subseteq} \mathbb{R}^n_-$. $f: U \to V$ is a **diffeomorphism** if f is bijective, differentiable, and has differentiable inverse.

Def. An n-dimensional **manifold with boundary** is a second-countable, Hausdorff (+ might as well throw in "connected", too) topological space that is locally homeomorphic to open subsets of $\mathbb{R}^n_-$ with differentiable transition functions.

Def. If M is an n-manifold with boundary, the boundary of M is $p \in M$ s.t. $\exists$ chart (U,h) at p s.t. $h(p) \in \partial h(U) \subseteq \mathbb{R}^n_-$.
The boundary of M is denoted ∂M.

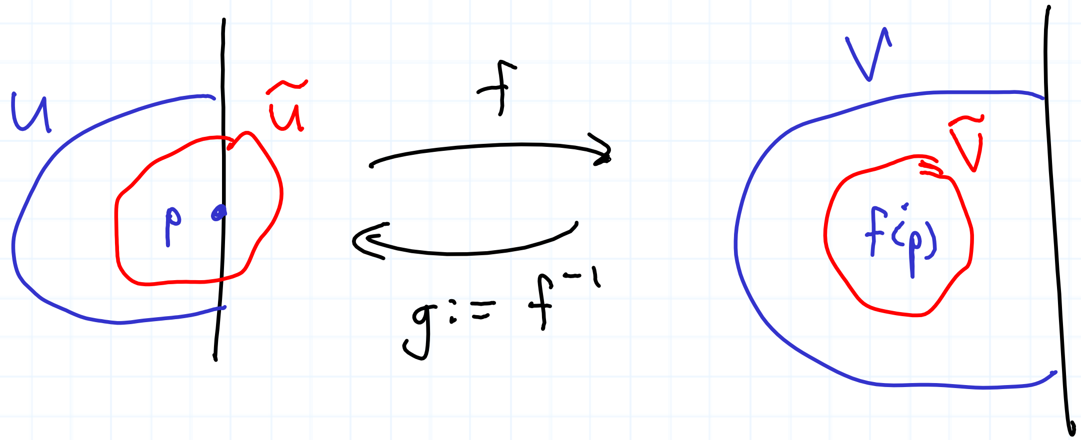
Prop. If $p \in \partial M$ and (U,h) is any chart at p, then $h(p) \in \partial h(U)$.

Pf/ This is a local question, handled by the following lemma.

Lemma Let $U, V \overset{\text{open}}{\subseteq} \mathbb{R}^n_-$ and let $f: U \to V$ be a diffeomorphism.
Then $f(\partial U) = \partial V$.

☆ See remarks on the last page before proceeding.

Pf/ Suppose $p \in \partial U$ and $f(p) \in V^\circ = V \setminus \partial U$ for sake of contradiction:



$\exists$ open $\hat{U} \subseteq \mathbb{R}^n$, a nbd. of p, and a function $\hat{f} : \hat{U} \to V$, differentiable in the usual sense, such that $\hat{f}|_U = f|_{\hat{U} \cap U}$.

Then $f(U \cap \hat{U}) \subseteq V$ is open in $\mathbb{R}^n_-$. Take $\hat{V} \subseteq f(U \cap \hat{U})$ a nbd. of $f(p)$, with $\hat{V}$ open in $\mathbb{R}^n$. This is ok since we are assuming $f(p) \notin \partial V$.

Then $\tilde{g} := g|_{\hat{V}} : \hat{V} \to \hat{U}$ is differentiable in the usual sense and $\hat{f} \circ \tilde{g} = id_{\hat{V}}$. Thus,

$$\tilde{f} \circ \hat{g} = id_{\tilde{V}} \Rightarrow (\tilde{f} \circ \hat{g})'_x = I \quad \forall x \in \tilde{V}$$

$$\Rightarrow \tilde{f}'_{\hat{g}(x)} \, \hat{g}'_x = I \quad \forall x \in \tilde{V}$$

$$\Rightarrow \hat{g}'_x \text{ invertible } \forall x \in \tilde{V}$$

$\Rightarrow \hat{g}'$ is locally a diffeomorphism on $\hat{V}$

$\Rightarrow \hat{g}$ is an open mapping ($\hat{g}$(open set) = open set)

$\Rightarrow \hat{g}(\hat{V}) \subseteq \tilde{U}$ is a nbd. of p, open in $\mathbb{R}^n$.

$\square$

Remarks. Let A, B be open subsets of $\mathbb{R}^n$.

① If $h: A \to B$ is a diffeomorphism and $U \subseteq A$ is open, then $h(U)$ is open.

Pf/ $h(U) = (h^{-1})^{-1}(U)$ and h^{-1} is continuous. $\square$

② If $h: A \to B$ is a $\underbrace{\text{local diffeomorphism}}$ and $U \subseteq A$ is open, then $h(U)$ is open.

i.e., $\forall p \in A$, $\exists V \overset{\text{open}}{\subseteq} A$ with $p \in V$ s.t. $h|V$ is a diffeomorphism onto its image.

③ Suppose $h: A \to B$ is differentiable and $\det(h'(x)) \neq 0 \ \forall x \in A$, then h is a local diffeomorphism.