Math 411

Properties of the integral \( M \) oriented \( n \)-manifold, \( w \in \text{\( \Omega \)}}^n \)

* If \( w \) is locally integrable, i.e., on an open neighborhood about each point, \( \text{\( \omega \)}} \) is integrable if \( \text{\( \omega \)}} \) has compact support, e.g., if \( M \) is compact.

* Let \( -M \) be \( M \) with the opposite orientation. Then \( \int_M w = -\int_{-M} w \).

* If \( f: M \rightarrow N \) is an orientation preserving diffeomorphism and \( w \in \text{\( \Omega \)}}^n \)

If integrable, then \( f^* w \) is integrable on \( M \) and

\[
\int_M f^* w = \int_N w
\]

** Proof:** Take a chart \( (U, h) \) at \( f(p) \) in \( N \). Then \( (f^{-1}(U), h \circ f) \) is a chart at \( p \).
\[ (f^*_w)_p \quad f^{-1}(U) \xrightarrow{f} U \xrightarrow{w_{f(p)}} \]

Coefficient of \( dx_{i_1} \wedge \cdots \wedge dx_{i_p} \) in the local expression for \((f^*_w)_p\):

\[
(f^*_w)_p \left( \left( \frac{\partial}{\partial x_{i_1}} \right)_p, \ldots, \left( \frac{\partial}{\partial x_{i_p}} \right)_p \right) = w_{f(p)} \left( f_*(\frac{\partial}{\partial x_{i_1}})_p, \ldots, f_*(\frac{\partial}{\partial x_{i_p}})_p \right)
\]

\[
= w_{f(p)} \left( \frac{\partial}{\partial y_{i_1}}|_{f(p)}, \ldots, \frac{\partial}{\partial y_{i_p}}|_{f(p)} \right)
\]

which is the coefficient of \( dy_{i_1}(f(p)) \wedge \cdots \wedge dy_{i_p}(f(p)) \) in the local expression of \( w_{f(p)} \). Hence, the corresponding local integrals will be equal.
Next goal: Stokes’ thm. \( \sum w = \sum dw \).

Manifolds with boundary (chapter 6)

Def. \( \mathbb{R}^n_- := \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \leq 0 \} \)

with the subspace topology inherited from \( \mathbb{R}^n \), i.e. the open subsets of \( \mathbb{R}^n_- \) are exactly sets of the form \( W \cap \mathbb{R}^n_- \) where \( W \) is an open subset of \( \mathbb{R}^n \).

Def. If \( U \subseteq \mathbb{R}^n_- \), then the boundary of \( U \) is
\[ \partial U = U \cap \{ x_1 = 0 \} = U \cap [\mathbb{R}^n \times \{ 0 \} \]
Def. (differentiability) Let $U \subseteq \mathbb{R}^n$ be open. A function $f : U \to \mathbb{R}^m$ is differentiable at $p \in U$ if $\exists$ open $W \subseteq \mathbb{R}^n$ with $p \in W$ and differentiable $g : W \to \mathbb{R}^m$ (in the usual sense) such that $f|_{W \cap U} = g|_{W \cap U}$.

Def. $U, V \subseteq \mathbb{R}^n$, $f : U \to V$ is a diffeomorphism if $f$ is bijective, differentiable, and has differentiable inverse.

Def. An $n$-dimensional manifold with boundary is a second-countable, Hausdorff (and might as well throw in “connected”, too) topological space that is locally homeomorphic to open subsets of $\mathbb{R}^n$ with differentiable transition functions.
Def. If $M$ is an $n$-manifold with boundary, the boundary of $M$ is $\partial M$ s.t. $\exists$ chart $(U,h)$ at $p$ s.t. $h(p) \in \partial h(U) \subseteq \mathbb{R}^n$. The boundary of $M$ is denoted $\partial M$.

Prop. If $p \in \partial M$ and $(U,h)$ is any chart at $p$, then $h(p) \in \partial h(U)$.

Proof. This is a local question, handled by the following lemma.

Lemma Let $U, V \subseteq \mathbb{R}^n$ and let $f: U \to V$ be a diffeomorphism. Then $f(\partial U) = \partial V$.

See remarks on the last page before proceeding.

Proof. Suppose $p \in \partial U$ and $f(p) \in V^0 = V \setminus \partial U$ for sake of contradiction.
\[ \exists \text{ open } \tilde{U} \subseteq \mathbb{R}^n, \text{ a nbd. of } p, \text{ and a function } \tilde{f} : \tilde{U} \to V, \] 

differentiable in the usual sense, such that \( \tilde{f}(\text{ln}) = f(\text{ln}). \)

Then \( f(\text{un} \tilde{U}) \subseteq V \) is open in \( \mathbb{R}^n. \) Take \( \tilde{V} \subseteq f(\text{un} \tilde{U}) \)
a nbd. of \( f(p), \) with \( \tilde{V} \) open in \( \mathbb{R}^n. \) This is OK since we are assuming \( f(p) \notin \partial V. \)

Then \( \tilde{g} = g|\tilde{V} : \tilde{V} \to \tilde{U} \) is differentiable in the usual sense and \( \tilde{f} \circ \tilde{g} = \text{id}\tilde{V}. \) Thus,
\[ \tilde{f} \circ \tilde{g} = \text{id}_{\tilde{V}} \implies (\tilde{f} \circ \tilde{g})'_x = I \quad \forall x \in \tilde{V} \]

\[ \implies \tilde{f}'_{\tilde{g}(x)} \circ \tilde{g}'_{x} = I \quad \forall x \in \tilde{V} \]

\[ \implies \tilde{g}'_{x} \text{ is invertible } \forall x \in \tilde{V} \]

\[ \implies \tilde{g}' \text{ is locally a diffeomorphism on } \tilde{V} \]

\[ \implies \tilde{g} \text{ is an open mapping } (\tilde{g}(\text{open set}) = \text{open set}) \]

\[ \implies \tilde{g} (\tilde{V}) \subseteq \tilde{U} \text{ is a nbhd. of } \tilde{p}, \text{ open in } \mathbb{R}^n. \]
Remarks. Let $A, B$ be open subsets of $\mathbb{R}^n$.

1. If $h : A \to B$ is a diffeomorphism and $U \subseteq A$ is open, then $h(U)$ is open.
   
   **Proof.** $h(U) = (h^{-1})^{-1}(U)$ and $h^{-1}$ is continuous. □

2. If $h : A \to B$ is a local diffeomorphism and $U \subseteq A$ is open, then $h(U)$ is open.
   
   i.e., $\forall p \in A$, $\exists V^\text{open} \subseteq A$ with $p \in V$ st. $h|V$ is a diffeomorphism onto its image.

3. Suppose $h : A \to B$ is differentiable and $\det(h'(x)) \neq 0$ $\forall x \in A$, then $h$ is a local diffeomorphism.