Oriented Manifolds

An oriented manifold is a manifold $M$ with a collection of orientations $\mathcal{O} = \{ \Theta_p \}_{p \in M}$ with $\Theta_p$ an orientation on $T_p M$ such that $\Theta$ is locally coherent, meaning that for each point at $M$, there is a chart $(U, h)$ at that point such that for all $p \in U$, the isomorphism induced by the chart:

$$T_p M \rightarrow \mathbb{R}^n$$

$$v \mapsto v(h(U, p))$$

takes $\Theta_p$ to the usual positive orientation of $\mathbb{R}^n$.

A diffeomorphism $f: M \rightarrow N$ is orientation preserving if $df_p: T_p M \rightarrow T_p N$ is orientation preserving for all $p \in M$. 

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A chart \((U, h)\) is orientation preserving if \(\forall p \in U\), the isomorphism
\[
T_p M \rightarrow \mathbb{R}^n
\]
takes \(h_p\) to the positive orientation of \(\mathbb{R}^n\).

An atlas \(\mathcal{U} = \{(U_i, h_i)\}\) is an orienting atlas if all the transition functions are orientation preserving, i.e., their derivatives have positive determinant. In this case, \(\exists!\) orientation of \(M\) such that \(\mathcal{U}\) consists of orientation preserving charts.

**Example**
- \(S^n\) is orientable
- a Möbius strip is not orientable
- \(\mathbb{P}^n\) is orientable iff \(n\) is odd (See HW.)
Integration on Manifolds

Basic idea: Let $M$ be an $n$-manifold, $w \in \Omega^n M$ (a section of $\Lambda^n T^* M$). Choosing coordinates $(U, h)$, we get a local expression for $w$:

$$w(p) = \sum \tilde{a}(p) \, dx_{i_1} \wedge \cdots \wedge dx_{i_n}.$$ 

1. If $A \subseteq U$, we let $\int_A w = \int_{h(A)} \tilde{a}$, where $a = \tilde{a} \circ h^{-1}$.

2. If $A \subseteq V$ for some other chart $(V, k)$, it turns out that this will get the same value for the integral!

3. To define $\int_M w$, divide $M$ up into nice disjoint pieces like $A$, above, which fit inside charts.

Note: The existence of an orientation must be relevant.