Math 411

Last time \( f : M \to \mathbb{R} \Rightarrow df_p : T_p M \to \mathbb{R} \), i.e. \( df \in T^*_p M \).

In local coords,

\[
(*) \quad df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p) \, dx^i \bigl( p \bigr).
\]

really \( \frac{\partial (f \circ h)}{\partial x_i}(p) \)

For general \( f : M \to N \) we get \( f_p^* : df_p : T_p M \to T_{f(p)} N \)

and, thus, \( f_p^* : df_p^* : T^*_{f(p)} N \to T^*_p M \) and

\( f_p^* : \Lambda^k T^*_{f(p)} N \to \Lambda^k T^*_p M \). In coords. \((U, h)\) at \( p \), \((V, h)\) at \( f(p) \) with \( U \subseteq \Phi^{-1}(V) \)!
Thus, \( df^*_p (dy_i) = \) the column of \( \left( \frac{df_i}{dx_j} (p) \right)^\text{tr} \)

\[ \begin{align*}
= & \quad \text{the row of} \quad \left( \frac{df_i}{dx_j} (p) \right) = \nabla f_i (p) \\
= & \quad \sum_j \frac{df_i}{dx_j} (p) dx_j = df_i, p \\
\end{align*} \]

\[ df^*_p (dy_i) = df_i, p \]

Thus, \( f^*_p : \bigwedge^k T^*N \rightarrow \bigwedge^k T^*_p M \) is given locally by

\[ f^*_p \left( \sum \omega^k (f(p)) \, dy_{i_1} \wedge \cdots \wedge dy_{i_k} \right) = \sum \omega^k (f(p)) \, df_{i_1} \wedge \cdots \wedge df_{i_k} . \]
Bundle

\[ f: M \to N \] induces mappings of bundles

\[ f_*: TM \to TN \quad f^*: \Omega^k N \to \Omega^k M \]

For instance, in local coords \((U, h), (V, k)\), as usual,

\[
\begin{align*}
\pi^*_m (U) & \xrightarrow{f_*} \pi^*_N (V) \\
\left( h(U) \times \mathbb{R}^m \right) & \xrightarrow{f} \left( k(V) \times \mathbb{R}^n \right) \\
\left( h(p), v \right) & \mapsto \left( k(f(p)), f^* p (v) \right)
\end{align*}
\]

**Point:** The mappings are defined locally (at each point \( p \), above) glue together
to give (smooth) mappings of bundles.
Example \( M = \mathbb{R}^2 \), \( N = \mathbb{R}^3 \)

\[
f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \\
(u, v) \mapsto (u^2 - v, u + 2v, v^2)
\]

\[
w_{xy}(x, y, z) \quad w_{yz}(x, y, z)
\]

\[
w = x^2 \, dx \land dy + (x + z) \, dy \land dz
\]

\[
\in \int_C \mathbb{R}^3 = \text{sections of } \wedge^2 \mathbf{T} \mathbb{R}^3
\]

\[
f^\star (w) = (u^2 - v) \, d(u^2 - v) \land d(u + 2v) + (u^2 - v + v^2) \, d(u + 2v) \land dv^2
\]

\[
= (u^2 - v) (2u \, du - dv) \land (du + 2dv) + (u^2 - v + v^2) (du + 2dv) \land (2vdv)
\]

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\[
= [(u^2 - v)(4u + 1) + 2(u^2 - v + v^2)v] \, du \land dv = \text{(etc)} \, du \land dv.
\]
Def. Let $V$ be a real vector space. Two ordered bases $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ have the same orientation if the mapping $V \to V$ where $v_i \mapsto w_i$ has positive determinant.

The property of having the same orientation defines an equivalence relation on the set of ordered bases for $V$ with two equivalence classes. Each equivalence class is called an orientation on $V$. Having chosen an orientation $\Theta$, we get an oriented vector space, $(V, \Theta)$. 
If an ordered basis \((w_1, \ldots, w_n)\) is in \(\Theta\), we say \((w_1, \ldots, w_n)\) is positively oriented; otherwise it's negatively oriented.

Example. \(V = \mathbb{R}^3\)

\[(e_1, e_2, e_3) \sim (e_2, e_3, e_1) \sim (e_3, e_1, e_2)\]

\[(e_2, e_1, e_3) \sim (e_1, e_3, e_2) \sim (e_3, e_2, e_1)\]

Also, for example, \((e_1 + e_2, e_2, e_3) \sim (e_1, e_2, e_3)\) since

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[\mathbb{R}^3 \xrightarrow{\sim} \mathbb{R}^3\]

has positive determinant.