Def. Let $\pi : T \to B$ be a surjection. A section of $\pi$ (or of $T$) is a mapping $s : B \to T$ such that $\pi \circ s = \text{id}_B$.

**VECTOR FIELDS**

Def. A section of the tangent bundle $TM \xrightarrow{\pi} M$ (differentially, of course) is called a vector field on $M$.

**Remarks**

1. Let $s : M \to TM$ be a vector field on $M$. The zero locus of $s$ is $\{ p \in M : s(p) = 0 \}$. Note that this is a well-defined concept, i.e., $s(p)$ being zero does not depend on the chart.

(Recall if $(U, h)$ is a chart at $p$ on $M$, we get a standard chart on $TM$ by

$\pi^{-1}(U) \xrightarrow{\tilde{h}} h(U) \times \mathbb{R}^n$

$v \in T_p M \xmapsto{} (h(q), v(U, h))$

If $v(U, h) = 0$, then $v(U, b) = 0$ for charts $(V, k)$ at $q$.}

\[\text{Math 411} \quad (\text{Start with a review. See p.7 of lecture 10.}) \quad 1\]
2. Let $S^2$ be a 2-sphere. If $TS^2 \cong S^2 \times \mathbb{R}^2$ as manifolds, then $TS^2$ would have a non-vanishing vector field. For instance, define $s : S^2 \to S^2 \times \mathbb{R}^2$. However, a famous theorem from topology (the hairy ball theorem) says there is no non-vanishing vector field on $S^2$.

$k$-forms

Def. A $k$-form on a manifold is a section of $\Lambda^k T^* M$.

The vector space of $k$-forms is denoted $\Lambda^k M$.

In coordinates: Let $(U, \phi)$ be a chart on $M$.

Then $T_p M \cong \mathbb{R}^n$ for each $p \in U$, give a basis $(\frac{\partial}{\partial x_i})_p$ for $T_p M$, a dual basis $dx_i^* : (\frac{\partial}{\partial x_i})_p \to \mathbb{R}$ for $\Lambda^1 T^*_p M$.
and a basis \( \delta_{n,p} \) for \( \Lambda^k T^*_p M \). Consider a k-form \( \omega \):

\[
\Lambda^k M \quad \omega_p \in \Lambda^k T^*_p M
\]

In local coordinates we get \( \omega(p) = \sum_i \omega_i(p) \, dx^i = \sum_i \omega_{i}(p,\ldots,p) \, dx^i \wedge \ldots \wedge dx^k \)

where each function \( \omega_i : \delta(U) \to \mathbb{R} \) is differentiable.

Mappings / Pullbacks

Let \( f : M \to \mathbb{R} \) on \( p \in M \). Take coords. \((U, h)\) at \( p \) on \( M \) and \((\mathbb{R} , id)\) on \( \mathbb{R} \).
We have a push forward mapping:

\[
\begin{array}{cccc}
& (\frac{\partial}{\partial x_i})_p & T_p M & f^*_p = df_p & T_{f(p)} \mathbb{R} \\
\downarrow & \downarrow & \downarrow & & \downarrow 12 \\
e_i & \mathbb{R}^n & Df(p) & \mathbb{R} & \frac{df}{dx_i}(p) := \frac{df}{dx_i}(h(p)) (k = id, here) \\
\end{array}
\]

Thus, \( df_p : T_p M \rightarrow \mathbb{R} \), i.e. \( df_p \in T^*_p M \). In coords,

\[
df_p = \sum_{j=1}^n \alpha_j(p) \, dx_j, p
\]

when \( \alpha_j(p) \) is determined by

\[
df_p \left( \left( \frac{\partial}{\partial x_i} \right)_p \right) = \sum_j \alpha_j(p) \, dx_j, p \left( \left( \frac{\partial}{\partial x_i} \right)_p \right) = \alpha_i(p).
\]

\[
\frac{df}{dx_i}(p) \quad \text{(from (A), above)} \quad \text{Thus,} \quad \star \quad df_p = \sum_{i=1}^n \frac{df}{dx_i}(p) \, dx_i, p.
\]

Example on \( \mathbb{R}^n \):

\[
d(\sqrt{x^2+y}) = \sqrt{2} \, dx + (2x + 1) \, dy
\]