Math 411

Exterior powers

Last time:

$$\begin{tikzcd} & \Lambda^\ell V \arrow[dr, dashed, "\exists!"] & \\ V^{\times \ell} \arrow[ur] \arrow[rr, "\text{alternating}"'] & & W \end{tikzcd}$$

Via this universal property, we identify alternating maps $V \times \cdots \times V \to W$ with linear maps $\Lambda^\ell V \to W$.

Remarks:

* If $e_1, \cdots, e_n$ is a basis for V, then $\{ e_{i_1} \wedge \cdots \wedge e_{i_\ell} : i_1 < \cdots < i_\ell \}$ is a basis for $\Lambda^\ell V$. Thus, $\dim_k \Lambda^\ell V = \binom{n}{\ell}$.

* $v_1, \ldots, v_\ell$ are linearly dependent iff $v_1 \wedge \cdots \wedge v_\ell = 0$.

Pf/ ($\Rightarrow$) Say $v_1 = \sum_{i=2}^{\ell} a_i v_i$. Then $v_1 \wedge \cdots \wedge v_\ell = \sum_{i=2}^{\ell} a_i (v_i \wedge \cdots \wedge v_\ell) = 0$.

($\Leftarrow$) Exercise. (See remark ✩✩, below.) $\square$

* It's tempting to identify the one dim'l space spanned by $v_1 \wedge \cdots \wedge v_\ell$ in $\Lambda^\ell V$ with the linear space spanned by $v_1, \dots, v_\ell$. More on this later.

** * Say $\dim V = n$. Let $e_1, \dots, e_n$ be a basis for V and let $v_1, \dots, v_n \in V$.

Thm $v_1 \wedge \cdots \wedge v_n = \det(v_1, \dots, v_n)\, e_1 \wedge \cdots \wedge e_n$.

Pf/ Define

$$\begin{aligned} D : V^{\times n} &\longrightarrow k \\ (v_1, \dots, v_n) &\longmapsto a \end{aligned}$$

where $v_1 \wedge \cdots \wedge v_n = a\; e_1 \wedge \cdots \wedge e_n$.

Then D is multilinear, alternating, and sends $(e_1, \dots, e_n) \mapsto 1$. $\square$

Products

$$\begin{aligned} \Lambda^\ell V \times \Lambda^m V &\longrightarrow \Lambda^{\ell+m} V \\ (\lambda, \mu) &\longmapsto \lambda \wedge \mu \end{aligned}$$

Algebra

$$\Lambda^\bullet V := \bigoplus_{l \geq 0} \Lambda^l V$$ where $\Lambda^0 V := k$ is a k-algebra. (the Grassmann algebra)

Symmetric Powers

$$\mathrm{Sym}^l := V^{\otimes l} \big/ T$$ where T is the linear span of expressions of the form $v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_l - v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_l$.

The image of $v_1 \otimes \cdots \otimes v_l$ in $\mathrm{Sym}^l V$ is denoted $v_1 \cdots v_l$.

Universal property

$$\begin{tikzcd}
 & \mathrm{Sym}^l V \arrow[dr, dashed, "\exists!"] & \\
V^{\times l} \arrow[ur] \arrow[rr, "\substack{\text{multilinear,} \\ \text{symmetric}}"'] & & W
\end{tikzcd}$$

Example

$$\begin{tikzcd} & \mathrm{Sym}^2\mathbb{R}^n \arrow[dr, dashed] & \\ \mathbb{R}^n\times\mathbb{R}^n \arrow[ur] \arrow[rr] & & \mathbb{R} \end{tikzcd} \qquad \begin{tikzcd} v\cdot w \arrow[d, mapsto] \\ \langle v,w\rangle \end{tikzcd}$$

$$(v,w) \longmapsto \langle v,w\rangle$$ (dot product)

Algebra $\quad \mathrm{Sym}\, V = \bigoplus_{\ell} \mathrm{Sym}^{\ell} V$

Basis If $e_1,\dots,e_n$ is a basis for V, then $\{e_1^{a_1}\cdots e_n^{a_n} : a_i \geq 0, \sum a_i = \ell\}$ is a basis for $\mathrm{Sym}^{\ell} V$.

Dual Space

Def. $V^* = \hom(V,k) =$ linear maps $V \to k$

For $L, M \in V^*$, $\lambda \in k$, and $v \in V$, we have $(\lambda L + M)(v) := \lambda L(v) + M(v)$.

Remarks.

- If $v_1, \ldots, v_n$ is a basis for V, define $v_1^*, \ldots, v_n^* \in V$ by

$$v_i^*(v_j) = \delta(i,j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}.$$

HW $v_1^*, \ldots, v_n^*$ is a basis for V^*.

Thus, $V \cong V^*$ if V is finite dimensional, but the isomorphism depends on a choice of basis.

* A linear mapping $L: V \to W$ induces $L^*: W^* \to V^*$
$$\varphi \mapsto \varphi \circ L$$

Functorial: $id_V: V \to V$ induces $id_{V^*}: V^* \to V^*$

$$\begin{tikzcd} V \arrow[r] \arrow[dr] & W \arrow[d] \\ & U \arrow[ul, phantom, "\circlearrowleft"] \end{tikzcd} \quad \text{induces} \quad \begin{tikzcd} V^* & W^* \arrow[l] \\ & U^* \arrow[ul] \arrow[u] \arrow[ul, phantom, "\circlearrowright"] \end{tikzcd}.$$

* An alternating bilinear mapping $\omega: V \times \cdots \times V \to \mathbb{R}$ is called an alternating form. Via the universal property it can be identified with a linear mapping $\omega: \Lambda^\ell V \to \mathbb{R}$ (abusing notation), hence, $\omega \in (\Lambda^\ell V)^*$

Example

An inner product $\langle\ ,\ \rangle: V \times V \to \mathbb{R}$ corresponds to $\mathrm{Sym}^2 V \to \mathbb{R}$, hence, to an element of $(\mathrm{Sym}^2 V)^*$.