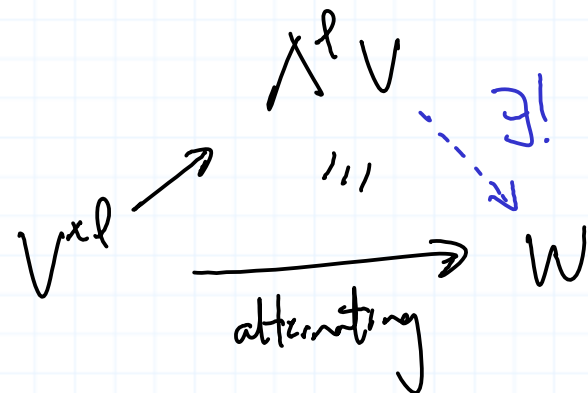


Math 411

Exterior powers

Last time:



①

Via this universal property, we identify alternating maps $V \times \dots \times V \rightarrow W$ with linear maps $\Lambda^l V \rightarrow W$.

Remarks:

* If e_1, \dots, e_n is a basis for V , then $\{e_{i_1} \wedge \dots \wedge e_{i_l} : i_1 < \dots < i_l\}$ is a basis for $\Lambda^l V$. Thus, $\dim_k \Lambda^l V = \binom{n}{l}$.

* v_1, \dots, v_l are linearly dependent iff $v_1 \wedge \dots \wedge v_l = 0$.

Pf/ (\Rightarrow) Say $v_1 = \sum_{i=2}^l a_i v_i$. Then $v_1 \wedge \dots \wedge v_l = \sum_{i=2}^l a_i (v_i \wedge \dots \wedge v_l) = 0$.

(\Leftarrow) Exercise. (See remark $\star \star$, below.) \square

②
* It's tempting to identify the one dim'l space spanned by $v_1 \wedge \dots \wedge v_n$ in $\wedge^n V$ with the linear space spanned by v_1, \dots, v_n . More on this later.

*
* Say $\dim V = n$. Let e_1, \dots, e_n be a basis for V and let $v_1, \dots, v_n \in V$.

Then $v_1 \wedge \dots \wedge v_n = \det(v_1, \dots, v_n) e_1 \wedge \dots \wedge e_n$.

Pf/ Define

$$D: V^{\times n} \longrightarrow k$$

$$(v_1, \dots, v_n) \mapsto a$$

where $v_1 \wedge \dots \wedge v_n = a e_1 \wedge \dots \wedge e_n$.

Then D is multilinear, alternating, and sends $(e_1, \dots, e_n) \mapsto 1$. \square

Products

$$\wedge^l V \times \wedge^m V \longrightarrow \wedge^{l+m} V$$

$$(\lambda, \mu) \mapsto \lambda \wedge \mu$$

Algebra

$$\Lambda^\bullet V := \bigoplus_{l \geq 0} \Lambda^l V$$

where $\Lambda^0 V := k$ is a k -algebra.
(the Grassmann algebra)

(3)

Symmetric Powers

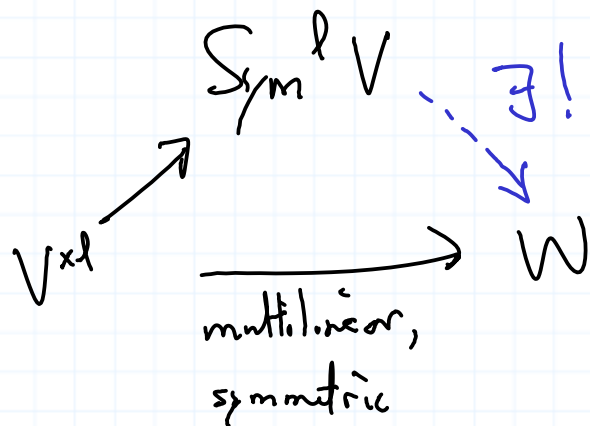
$$\text{Sym}^l := \frac{V^{\otimes l}}{T}$$

where T is the linear span of

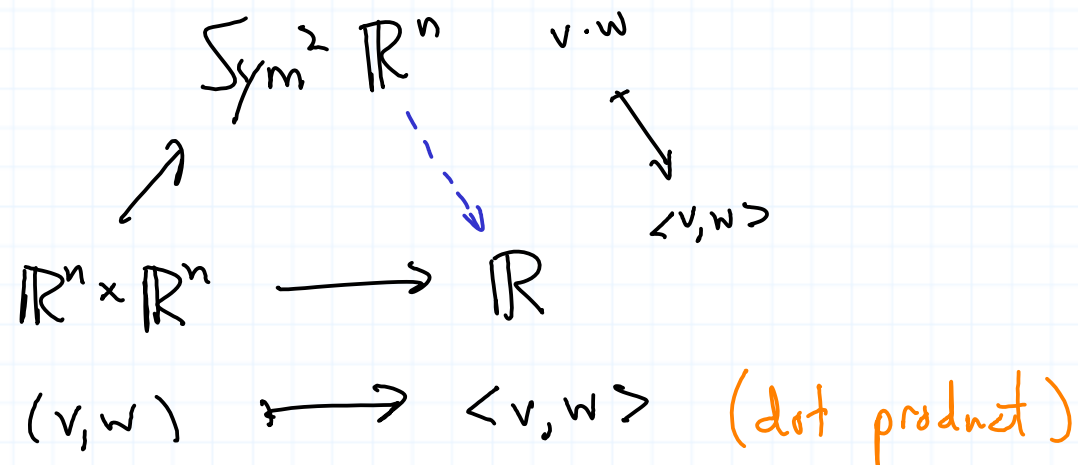
expressions of the form $v_1 \otimes \dots \otimes v_i \otimes \dots \otimes v_j \otimes \dots \otimes v_l - v_1 \otimes \dots \otimes v_j \otimes \dots \otimes v_i \otimes \dots \otimes v_l$.

The image of $v_1 \otimes \dots \otimes v_l$ in $\text{Sym}^l V$ is denoted $v_1 \cdots v_l$.

Universal property



Example



Algebra

$$\text{Sym } V = \bigoplus_l \text{Sym}^l V$$

Basis

If e_1, \dots, e_n is a basis for V , then $\{e_1^{a_1} \dots e_n^{a_n} : a_i \geq 0, \sum a_i = l\}$ is a basis for $\text{Sym}^l V$.

⑤

Dual Space

Def. $V^* = \text{hom}(V, k) =$ linear maps $V \rightarrow k$

For $L, M \in V^*$, $\lambda \in k$, and $v \in V$, we have $(\lambda L + M)(v) := \lambda L(v) + M(v)$.

Remarks.

* If v_1, \dots, v_n is a basis for V , define $v_1^*, \dots, v_n^* \in V^*$ by

$$v_i^*(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

HW v_1^*, \dots, v_n^* is a basis for V^* .

Thus, $V \cong V^*$ if V is finite dimensional, but the isomorphism depends on a choice of basis.

* A linear mapping $L: V \rightarrow W$ induces $L^*: W^* \rightarrow V^*$

(6)

Functorial: $\text{id}_V: V \rightarrow V$ induces $\text{id}_{V^*}: V^* \rightarrow V^*$

$$\varphi \mapsto \varphi \circ L$$

$$\begin{array}{ccc} V \rightarrow W & \text{induces} & V^* \leftarrow W^* \\ \downarrow \text{"} \downarrow & & \uparrow \text{"} \uparrow \\ U & & U^* \end{array}$$

* An alternating bilinear mapping $w: V \times \dots \times V \rightarrow \mathbb{R}$ is called an **alternating form**. Via the universal property it can be identified with a linear mapping $w: \Lambda^l V \rightarrow \mathbb{R}$ (abusing notation), hence, $w \in (\Lambda^l V)^*$

Example

An inner product $\langle, \rangle: V \times V \rightarrow \mathbb{R}$ corresponds to $\text{Sym}^2 V \rightarrow \mathbb{R}$, hence, to an element of $(\text{Sym}^2 V)^*$.