Math 411

Universal property of the tensor product.

Let V, W, U be vector spaces over k.

Def. $f : V \times W \to U$ is bilinear if it is linear with respect to each variable:

$$f(\lambda v + v', w) = \lambda f(v,w) + f(v', w)$$

$$f(v, \lambda w + w') = \lambda f(v,w) + f(v,w') \qquad \forall \lambda \in k;\ v,v' \in V;\ w,w' \in W.$$

The tensor product satisfies:

$$\begin{tikzcd} & V \otimes W \arrow[dr, dashed, "\exists!\, g"] & \\ V \times W \arrow[ur, "\iota"] \arrow[rr, "f"'] \arrow[urr, phantom, "/\!/"] & & U \end{tikzcd}$$

Thus, using the tensor product, we represent bilinear maps like f with linear maps like g.

where $\iota : V \times W \to V \otimes W$ is bilinear,
$(v,w) \mapsto v \otimes w$

$f : V \times W \to U$ is any bilinear mapping,

and $g : V \otimes W \to U$ is linear, making the diagram commute.

The function g is defined by $g(v \otimes w) = f(v,w)$ and extended linearly.

(extending by linearity to all of $F(V,W)$)

Pf/ Exercise. For instance, to show g is well defined, define $G : F(V,W) \to U$ by $G([v,w]) = f(v,w)$, then check $G(T) = 0$. For instance, $G([v+v',w] - [v,w] - [v',w]) =$
$G([v+v',w]) - G([v,w]) - G([v',w])$ (by definition of G)
$= f(v+v',w) - f(v,w) - f(v',w) = 0$, since f is bilinear. $\square$

Examples of the use of the universal property for tensor products:

Prop. $V \otimes k \simeq V$

Pf/ Define $f : V \to V \otimes k$ by $f(v) = v \otimes 1$. This gives a linear mapping.

To define the inverse, note that
$$\begin{aligned} V \times k &\to V \\ (v, \alpha) &\mapsto \alpha v \end{aligned}$$
is bilinear. Hence, the universal property guarantees the linear mapping
$$\begin{aligned} g : V \otimes k &\to V \\ v \otimes \alpha &\mapsto \alpha v \end{aligned}.$$
It's easy to check $f \circ g = \mathrm{id}_{V \otimes k}$ and $g \circ f = \mathrm{id}_V$. $\square$

Similarly, one may show:

Prop. 1) $(V \otimes W) \otimes U \simeq V \otimes (W \otimes U)$
$$(v \otimes w) \otimes u \longmapsto v \otimes (w \otimes u)$$

2) $V \otimes (W \oplus U) \simeq (V \otimes W) \oplus (V \otimes U)$.
$$v \otimes (w, u) \longmapsto (v \otimes w, v \otimes u)$$

Pf/ We'll just prove 2, since 1 is similar. Define

$$f : V \times (W \oplus U) \to (V \otimes W) \oplus (V \otimes U)$$
$$(v, (w, u)) \mapsto (v \otimes w, v \otimes u)$$

It is easy to check that f is bilinear, hence, induces a linear mapping $\tilde{f} : V \otimes (W \oplus U) \to (V \otimes W) \oplus (V \otimes U)$.

$$v \otimes (w, u) \mapsto (v \otimes w, v \otimes u)$$

Similarly, the bilinear mappings $g_1 : V \times W \to V \otimes (W \oplus U)$
$$(v, w) \mapsto v \otimes (w, 0)$$
and $g_2 : V \times U \to V \otimes (W \oplus U)$
$$(v, u) \mapsto v \otimes (0, u)$$
induce mappings from $V \otimes W$ and $V \otimes U$ to $V \otimes (W \oplus U)$.

By the universal property of the coproduct, we get the linear mapping $\tilde{g} : (V \otimes W) \oplus (V \otimes U) \longrightarrow V \otimes (W \oplus U)$

$$(v \otimes w, v' \otimes u) \mapsto v \otimes (w, 0) + v' \otimes (0, u).$$

It's straightforward to check that $\tilde{f}$ and $\tilde{g}$ are inverses. $\square$

Part 2 allows one to conclude that if V has basis $v_1, \dots, v_m$ and W has basis $w_1, \dots, w_n$, then $V \otimes W$ has basis $\{v_i \otimes w_j\}_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$:

$$V \otimes W \simeq V \otimes (\overbrace{k \oplus k \oplus \cdots \oplus k}^{n\text{-times}}) \simeq \bigoplus_{i=1}^{n} (V \otimes k) \simeq \bigoplus_{i=1}^{n} V \simeq \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} k \simeq k^{mn}$$

with the $v_i \otimes w_j$ being sent to the standard basis vectors of k^{mn}.

Define $V_1 \otimes \cdots \otimes V_n = V_1 \otimes (V_2 \otimes (\cdots \otimes (V_{n-1} \otimes V_n)) \cdots)$ (c.f. ①, above)

to get a vector space satisfying the universal property

$$\begin{array}{ccc} & V_1 \otimes \cdots \otimes V_n & \\ \nearrow & & \searrow \exists g \\ V_1 \times \cdots \times V_n & \xrightarrow[\text{multilinear}]{f} & U \end{array}$$

$(v_1, \ldots, v_n) \mapsto v_1 \otimes \cdots \otimes v_n$

(i.e. linear w.r.t. each coordinate)

Exterior Powers

$$\Lambda^l V = V \otimes \cdots \otimes V \,/\, T$$

where $T = \operatorname{span}\{ v_1 \otimes \cdots \otimes v_l : v_i = v_j$ for some $i \neq j \}$.

The equivalence class of $v_1 \otimes \cdots \otimes v_l$ is denoted $v_1 \wedge \cdots \wedge v_l$.

Note: we could use the property from the following proposition to define T in the case $\mathrm{char}(k) \neq 2$.

Prop. $v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_\ell = - v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_\ell$.

Pf/ $$0 = v_1 \wedge \cdots \wedge (v_i + v_j) \wedge \cdots \wedge (v_i + v_j) \wedge \cdots \wedge v_\ell$$
$$= \text{etc.}$$
$$= v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_\ell + v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_\ell .$$
$\square$

Universal property $\quad f: V \times \cdots \times V \to W$

is alternating if multilinear and $f(v_1, \ldots, v_\ell) = 0$ if $v_i = v_j$ for some $i \neq j$.
(Thus, $f(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_\ell) = - f(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_\ell)$ by the same argument as in the proposition.)

$$\begin{tikzcd}
 & \Lambda^\ell V \arrow[dr, dashed, "\exists!"] & \\
V^{\times \ell} \arrow[ur] \arrow[rr, "\text{alternating}"'] & & W
\end{tikzcd}$$