

Universal property of the tensor product.

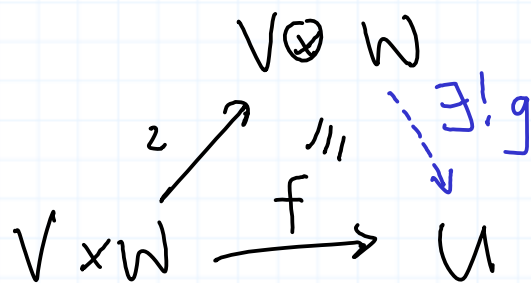
Let V, W, U be vector spaces over k .

Def. $f: V \times W \rightarrow U$ is **bilinear** if it is linear with respect to each variable:

$$f(\lambda v + v', w) = \lambda f(v, w) + f(v', w)$$

$$f(v, \lambda w + w') = \lambda f(v, w) + f(v, w') \quad \forall \lambda \in k; v, v' \in V; w, w' \in W.$$

The tensor product satisfies:



Thus, using the tensor product, we represent bilinear maps like f with linear maps like g .

where $z: V \times W \rightarrow V \otimes W$ is bilinear,
 $(v, w) \mapsto v \otimes w$

(2)

$f: V \times W \rightarrow U$ is any bilinear mapping,

and $g: V \otimes W \rightarrow U$ is linear, making the diagram commute.

The function g is defined by $g(v \otimes w) = f(v, w)$ and extended linearly.

Pf/ Exercise. For instance, to show g is (well defined, define $G: F(V, W) \rightarrow U$ by $G([v, w]) = f(v, w)$, then check $G(T) = 0$. For instance, $G([v+v', w] - [v, w] - [v', w]) = G([v+v', w]) - G([v, w]) - G([v', w])$ (by definition of G)
 $= f(v+v', w) - f(v, w) - f(v', w) = 0$, since f is bilinear. \square

(extending by linearity to all of $F(V, W)$)

Examples of the use of the universal property for tensor products:

③

Prop. $V \otimes k \cong V$

Pf/ Define $f: V \rightarrow V \otimes k$ by $f(v) = v \otimes 1$. This gives a linear mapping.

To define the inverse, note that $V \times k \rightarrow V$ is bilinear. Hence,
 $(v, \alpha) \mapsto \alpha v$

the universal property guarantees the linear mapping $g: V \otimes k \rightarrow V$,
 $v \otimes \alpha \mapsto \alpha v$

It's easy to check $f \circ g = \text{id}_{V \otimes k}$ and $g \circ f = \text{id}_V$. \square

Similarly, one may show:

Prop. ① $(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$

$$(v \otimes w) \otimes u \mapsto v \otimes (w \otimes u)$$

② $V \otimes (W \oplus U) \cong (V \otimes W) \oplus (V \otimes U)$.

$$v \otimes (w, u) \mapsto (v \otimes w, v \otimes u)$$

Pf/ We'll just prove ②, since ① is similar. Define

④

$$f: V \times (W \oplus U) \rightarrow (V \otimes W) \oplus (V \otimes U)$$
$$(v, (w, u)) \mapsto (v \otimes w, v \otimes u)$$

It is easy to check that f is bilinear, hence, induces a linear mapping

$$\tilde{f}: V \otimes (W \oplus U) \rightarrow (V \otimes W) \oplus (V \otimes U)$$
$$v \otimes (w, u) \mapsto (v \otimes w, v \otimes u)$$

Similarly, the bilinear mappings $g_1: V \times W \rightarrow V \otimes (W \oplus U)$

$$(v, w) \mapsto v \otimes (w, 0)$$

and $g_2: V \times U \rightarrow V \otimes (W \oplus U)$

$$(v, u) \mapsto v \otimes (0, u)$$

induce mappings from $V \otimes W$ and $V \otimes U$ to $V \otimes (W \oplus U)$.

By the universal property of the coproduct, we get the linear

mapping $\tilde{g}: (V \otimes W) \oplus (V \otimes U) \rightarrow V \otimes (W \oplus U)$ ⑤

$$(v \otimes w, v' \otimes u) \mapsto v \otimes (w, 0) + v' \otimes (0, u),$$

It's straightforward to check that \tilde{f} and \tilde{g} are inverses, \square

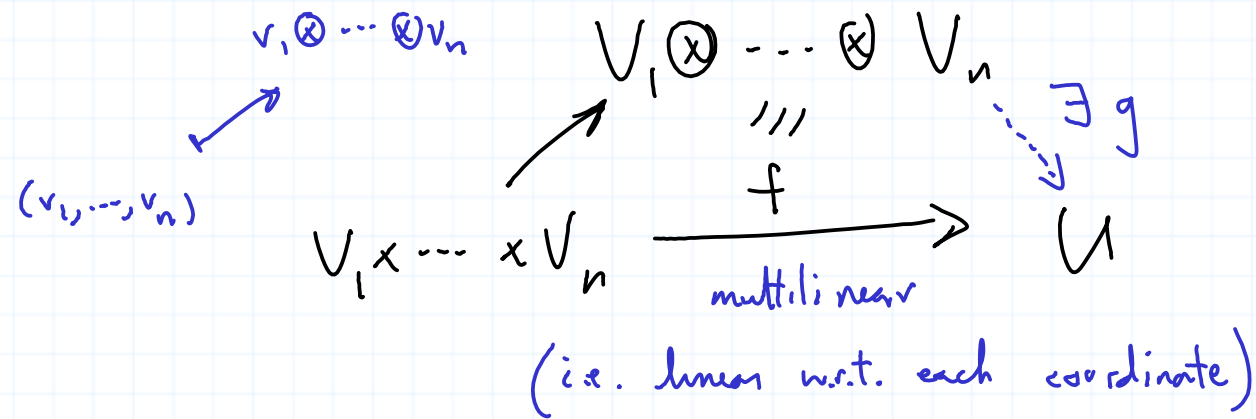
Part ② allows one to conclude that if V has basis v_1, \dots, v_m and W has basis w_1, \dots, w_n , then $V \otimes W$ has basis $\{v_i \otimes w_j\}_{\substack{i=1, \dots, m \\ j=1, \dots, n}}$

$$V \otimes W \cong V \otimes (\overbrace{k \oplus k \oplus \dots \oplus k}^{n \text{ - times}}) \cong \bigoplus_{i=1}^n (V \otimes k) \cong \bigoplus_{i=1}^n V \cong \bigoplus_{i=1}^n \bigoplus_{j=1}^m k \cong k^{mn}$$

with the $v_i \otimes w_j$ being sent to the standard basis vectors of k^{mn} .

Define $V_1 \otimes \dots \otimes V_n = V_1 \otimes (V_2 \otimes (\dots \otimes (V_{n-1} \otimes V_n) \dots))$ (c.f. ①, above) ⑥

to get a vector space satisfying the universal property



Exterior Powers

$$\Lambda^l V = \frac{V \otimes \dots \otimes V}{T} \quad \text{where } T = \text{span} \{ v_i \otimes \dots \otimes v_l : v_i = v_j \text{ for some } i \neq j \}.$$

The equivalence class of $v_1 \otimes \dots \otimes v_l$ is denoted $v_1 \wedge \dots \wedge v_l$.

Note: we could use the property from the following proposition to define T in the case $\text{char}(k) \neq 2$.

Prop. $v_1 \wedge \dots \wedge v_i \wedge \dots \wedge v_j \wedge \dots \wedge v_\ell = -v_1 \wedge \dots \wedge v_j \wedge \dots \wedge v_i \wedge \dots \wedge v_\ell$.

Pf/ $0 = v_1 \wedge \dots \wedge (v_i + v_j) \wedge \dots \wedge (v_i + v_j) \wedge \dots \wedge v_\ell$
 $=$ etc.
 $= v_1 \wedge \dots \wedge v_i \wedge \dots \wedge v_j \wedge \dots \wedge v_\ell + v_1 \wedge \dots \wedge v_j \wedge \dots \wedge v_i \wedge \dots \wedge v_\ell$. \square

Universal property $f: V \times \dots \times V \rightarrow W$

is **alternating** if multilinear and $f(v_1, \dots, v_\ell) = 0$ if $v_i = v_j$ for some $i \neq j$.
(Thus, $f(v_1, \dots, v_i, \dots, v_j, \dots, v_\ell) = -f(v_1, \dots, v_j, \dots, v_i, \dots, v_\ell)$ by the same argument as in the proposition.)

