Math 411

Tangent Space

M manifold, $p \in M$.

Version 1 (Geometric) From last time:

$$K_p = \{ \alpha : (-\varepsilon, \varepsilon) \to M : \varepsilon > 0,\ \alpha \in C^\infty,\ \alpha(0) = p \}$$

Equivalence relation on K_p: $\alpha \sim \beta$ if $(h \circ \alpha)'(0) = (h \circ \beta)'(0)$
for some chart (U, h) at p with $h(p) = 0 \in \mathbb{R}^n$

$$T_p^{\text{geom}}(M) = K_p / \sim$$

Version 2 (Algebraic) $T_p^{\text{alg}}(M)$ = derivations of germs of functions at p.

From last time: $\mathcal{E}_p = C^\infty$ germs at p.

A tangent vector is a linear mapping:

$$v : \mathcal{E}_p \longrightarrow \mathbb{R}$$

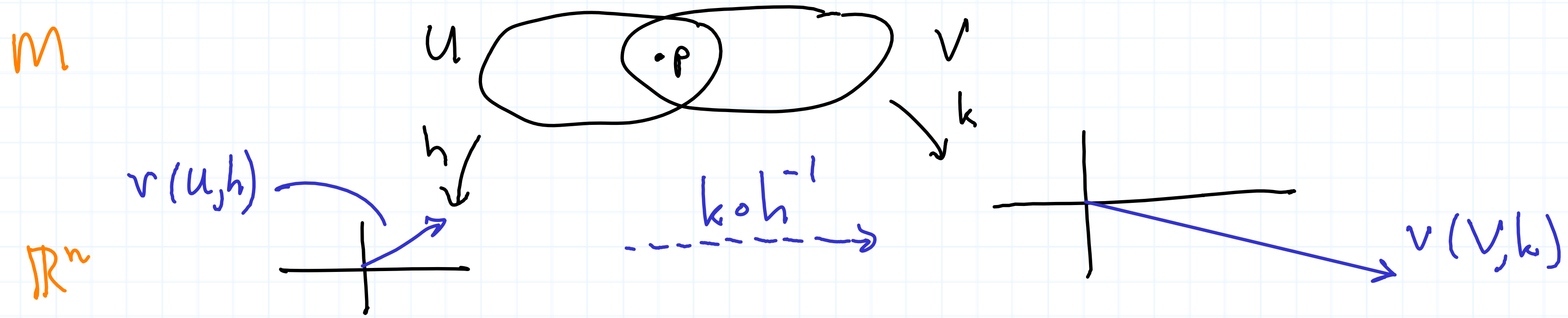
satisfying the product rule:

$$v(fg) = v(f)\,g(p) + f(p)\,v(f).$$

Version 3 (Physical) $\quad T_p^{phys}(M) = \dfrac{\text{charts} \times \text{vectors}}{\sim \text{ derivatives of transition functions}}$

Let $D_p(M)$ = charts at p. A **tangent vector** at p is a mapping $v : D_p(M) \to \mathbb{R}^n$ with the following property:

if (U,h), (V,k) are charts at p, then $(k \circ h^{-1})'\big(v(U,h)\big) = v(V,k)$.



Notation Continuing from above, define $x^i := h_i$, the i^{th} component of h. Similarly, define $\tilde{x}^i = k_i$, and $v^i = v(U,h)_i$, and $\tilde{v}^i = v(V,k)_i$.

In terms of the Jacobian matrix, we want

We may assume $h(p)=0$, for convenience.

$$\left[J(k\circ h^{-1})(0)\right] v(U,h) = \underbrace{\left[\frac{\partial \tilde{x}^i}{\partial x^j}\right]}_{\text{Notation denoting the partials of the components of } k\circ h^{-1}} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \begin{bmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{bmatrix},$$

i.e. $\displaystyle\sum_j \frac{\partial \tilde{x}^i}{\partial x^j} v^j = \tilde{v}^i$, or in physics notation: $\tilde{v}^\mu = \underbrace{\frac{\partial \tilde{x}^\mu}{\partial x^\nu} x^\nu}_{\text{Einstein sum notation}}$

Goal: Show these three versions of tangent space are equivalent (linearly isomorphic)

$$T_p^{geom}(M) \longrightarrow T_p^{alg}(M)$$

$[\alpha]$ $\qquad\qquad$ $v_{[\alpha]} : \xi_p \longrightarrow \mathbb{R}$

$\alpha : (-\varepsilon, \varepsilon) \to M$ $\qquad\qquad$ $[f] \longmapsto (f \circ \alpha)'(0)$

Check

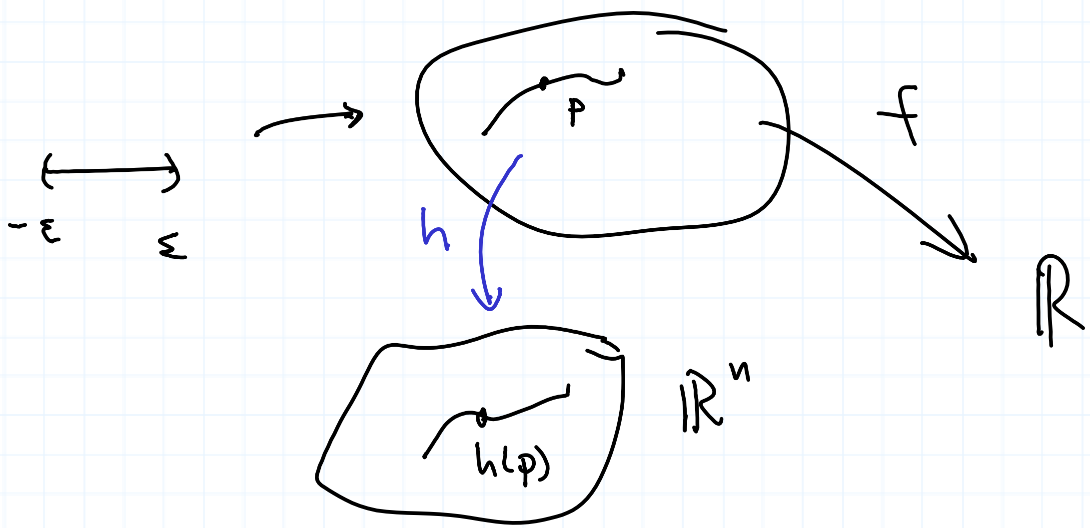
1. Well-defined
   - choice of representative f? ✓
   - choice of representation α?

We may assume same ε for α and β

Say $\alpha \sim \beta$. Then $\alpha, \beta : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$, $\alpha(0) = \beta(0) = p$

$$(h \circ \alpha)'(0) = (h \circ \beta)'(0).$$

Thm $(f\circ\alpha)'(0) = (f\circ h^{-1}\circ h\circ\alpha)'(0) = (f\circ h^{-1})'(h(p))\,(h\circ\alpha)'(0)$



$$= (f\circ h^{-1})'(h(p))\,(h\circ\beta)'(0)$$

$$= (f\circ h^{-1}\circ h\circ\beta)'(0)$$

$$= (f\circ\beta)'(0),$$

by the chain rule.

2. $v_{[\alpha]}$ is a derivation.

$$v_{[\alpha]}(fg) = ((fg)\circ\alpha)'(0) = ((f\circ\alpha)\cdot(g\circ\alpha))'(0) = [(f\circ\alpha)'(g\circ\alpha) + (f\circ\alpha)(g\circ\alpha)'](0)$$

product composition product ordinary product rule from 1-var. calc.

$$= (f\circ\alpha)'(0)\,g(\alpha(0)) + f(\alpha(0))\,(g\circ\alpha)'(0) = v_{[\alpha]}(f)\,g(p) + f(p)\,v_{[\alpha]}(g).$$

3. linearity. First, what is the linear structure on $T_p^{geom}(M)$?
Fix a chart (U,h) at p, and define

$$\begin{aligned} \Phi : T_p M &\longrightarrow \mathbb{R}^n \\ [\alpha] &\longmapsto (h\circ\alpha)'(0) \end{aligned}$$

Then Φ is a bijection (check!), so we can use it to induce a linear structure on $T_p(M)$:

$$\lambda[\alpha] + [\beta] := \Phi^{-1}(\lambda\,\Phi(\alpha) + \Phi(\beta)).$$

Exercise: check that this linear structure is independent of the choice of chart. Note: $\Phi^{-1}(\lambda\Phi(\alpha)+\Phi(\beta))$ is represented by any curve $\gamma \in K_p$ such that $(h\circ\gamma)'(0) = \lambda(h\circ\alpha)'(0) + (h\circ\beta)'(0)$.

We would like to show $v : T_p^{geom}(M) \to T_p^{alg}(M)$ is linear. Take $[\alpha], [\beta] \in T_p^{geom}(M)$. Then

$$v_{\lambda[\alpha]+[\beta]}(f) = (f\circ\gamma)'(0) \quad \text{where } (h\circ\gamma)'(0) = \lambda(h\circ\alpha)'(0) + (h\circ\beta)'(0)$$

having chosen a chart (U,h). Then,

$$(f\circ\gamma)'(0) = (f\circ h^{-1} h\gamma)'(0) = (fh^{-1})'(h\gamma(0))(h\gamma)'(0)$$

$$= (fh^{-1})'(h(p))\left[\lambda(h\circ\alpha)'(0) + (h\circ\beta)'(0)\right]$$

$$= \lambda(fh^{-1})'(h(p))(h\circ\alpha)'(0) + (f\circ h^{-1})'(h(p))(h\circ\beta)'(0)$$

$$= \lambda(fh^{-1}h\alpha)'(0) + (fh^{-1}h\beta)'(0)$$

$$= \lambda(f\circ\alpha)'(0) + (f\circ\beta)'(0)$$

$$= \lambda v_\alpha f + v_\beta f. \qquad \square$$