Math 411

$\mathbb{P}^2$ = lines through origin in $\mathbb{R}^2$ $= \mathbb{R}^3 \setminus \{(0,0,0)\} \big/ (x,y,z) \sim \lambda(x,y,z)$ for $\lambda \neq 0$.

Standard charts

$U_x = \{(x,y,z) \in \mathbb{P}^2 : x \neq 0\}$, U_y = etc., U_z = etc.

$\varphi_x : U_x \longrightarrow \mathbb{R}^2$, φ_y = etc., φ_z = etc.

$(x,y,z) \longmapsto \left(\frac{y}{x}, \frac{z}{x}\right)$

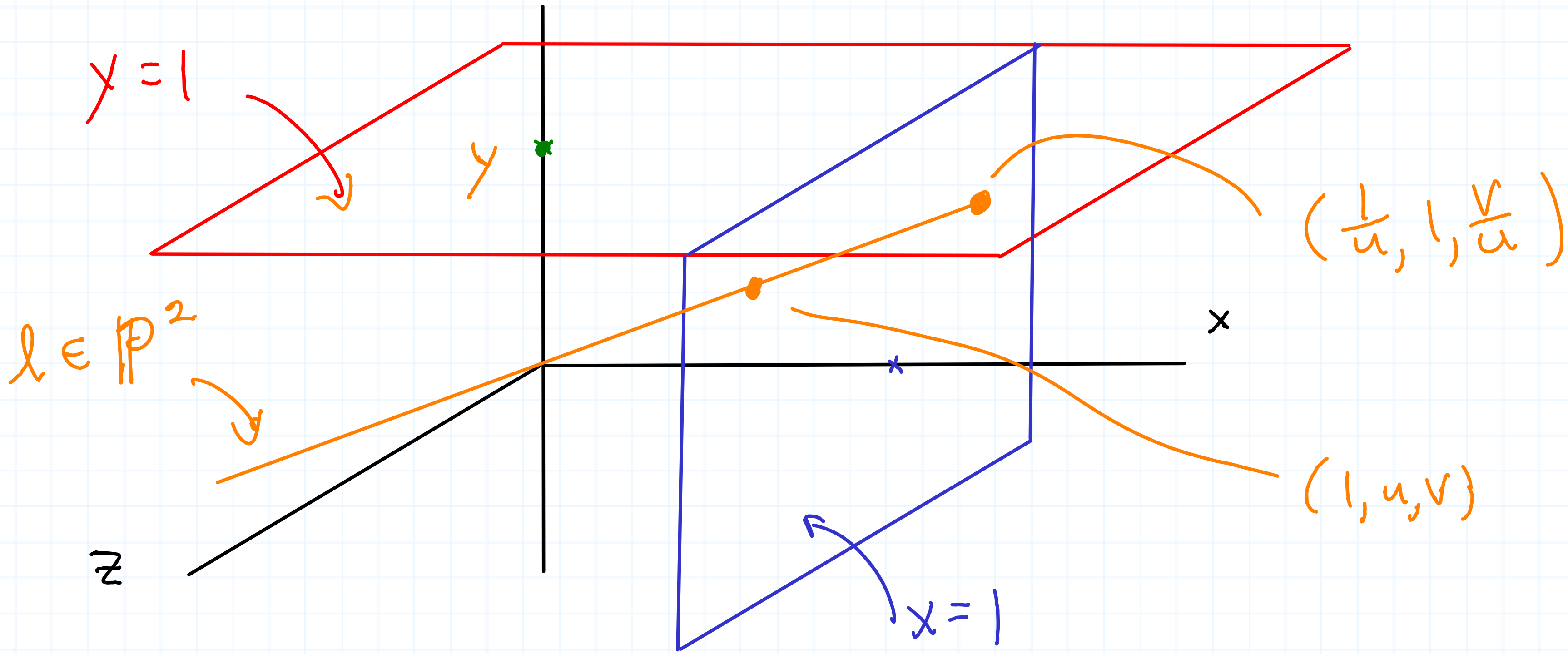
[inverse: $(1,a,b) \longleftarrow (a,b)$]

Standard atlas $\mathcal{U} = \{(U_x,\varphi_x),(U_y,\varphi_y),(U_z,\varphi_z)\}$

Transition functions

$$
\begin{tikzcd}
 & (1,u,v) \arrow[ddr, bend left, "\varphi_y"] & \\
 & U_x \cap U_y \arrow[dl, "\varphi_x"'] \arrow[dr, "\varphi_y"] & \\
\mathbb{R}^2 \arrow[rr, "\varphi_y \varphi_x^{-1}"] & & \mathbb{R}^2 \\
(u,v) \arrow[uuur, bend left, "\varphi_x^{-1}"] \arrow[rr] & & \left(\frac{1}{u}, \frac{v}{u}\right)
\end{tikzcd}
$$

NOTE $\varphi_y \circ \varphi_x^{-1}$ is smooth since it's a quotient of polynomials



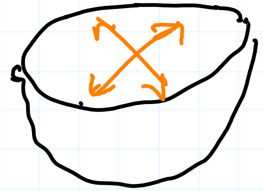
More possible charts: Fix any plane $\Pi \subseteq \mathbb{R}^3$, not containing the origin. Then

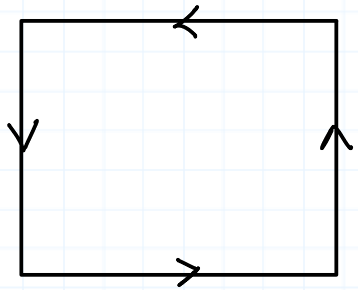
$$\{\ell \text{ not parallel to } \Pi\} \longrightarrow \Pi \cong \mathbb{R}^2$$
$$\ell \longmapsto \ell \cap \Pi$$

Lines in $\mathbb{P}^2$ A line in $\mathbb{P}^2$ is a plane through the origin in $\mathbb{R}^3$, i.e., a 2-dim'l vector subspace of $\mathbb{R}^3$.

HW for next week: Show that 2 ~~pts.~~ distinct points in $\mathbb{P}^2$ determine a unique line and two distinct lines in $\mathbb{P}^2$ determine a unique point.

Another model for $\mathbb{P}^2$

$\mathbb{P}^2 = S^2 / {x \sim -x}$ or  hemisphere with "zipped up" boundary.

or  (glue opposite sides with twists, as indicated by arrows).

$\mathbb{P}^n$

In general $\mathbb{P}^n$ = lines through origin in $\mathbb{R}^{n+1}$

= 1 dim'l subspaces of $\mathbb{R}^{n+1}$

Standard open cover: $U_i = \{ x \in \mathbb{P}^n : x_i \neq 0 \}$

harts: $\varphi_i : U_i \longrightarrow \mathbb{R}^n$

$$(x_0, x_1, \dots, x_n) \longmapsto \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i} \right)$$

omit i^{th} coord.

transition functions: HW for next week.

Manifolds

M topological space

An *n-dim'l chart at $p \in M$* is an open set U_p containing p and a homeomorphism $h: U_p \to \Omega$ onto some open subset $\Omega \subseteq \mathbb{R}^n$.

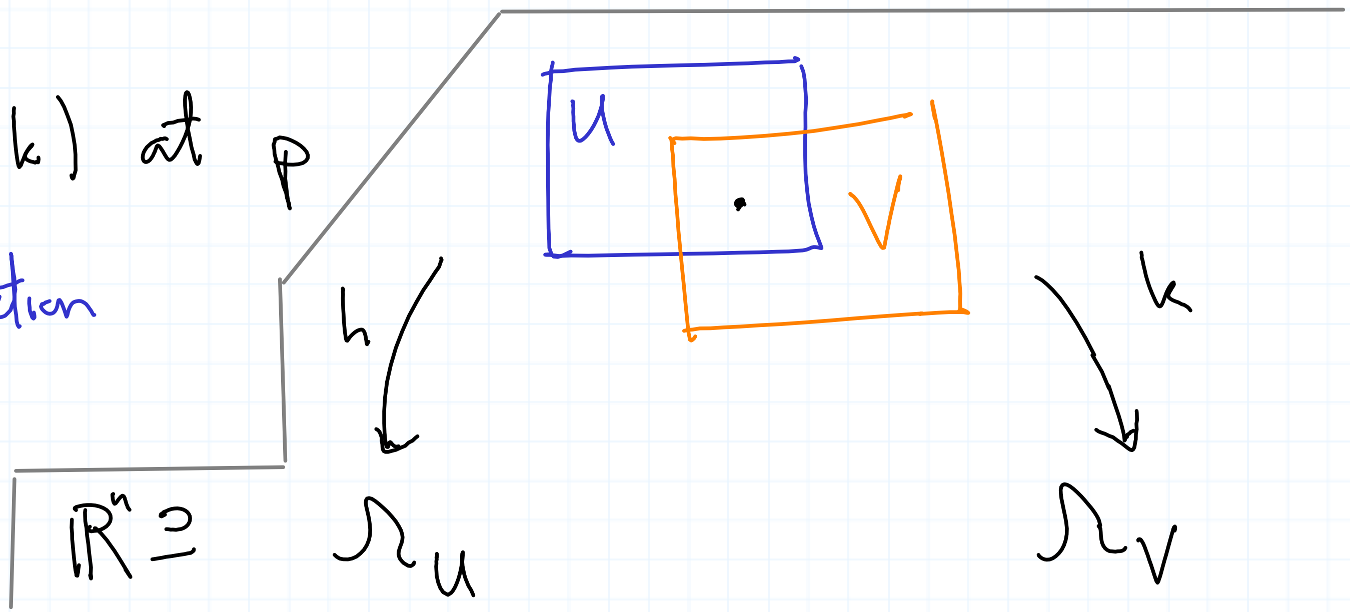
The set M is *locally Euclidean* if there exists an n-dim'l chart each point of M.

Given charts (U,h), (V,k) at p

the corresponding *transition function*

is

$$k \circ h^{-1} : h(U \cap V) \to k(U \cap V)$$



The charts (U,h) and (V,k) are differentially related if $k \circ h^{-1}$ is a diffeomorphism, i.e. if $k \circ h^{-1}$ and its inverse are both smooth (i.e., C^∞). an equivalence relation

A collection of n-dim'l charts, $\mathcal{U} = \{(U_i, \varphi_i)\}$, for M is an n-dim'l atlas for M if $\bigcup_i U_i = M$.

If $\mathcal{U}, \mathcal{B}$ are atlases for M say $\mathcal{U} \sim \mathcal{B}$ if the charts of $\mathcal{U} \cup \mathcal{B}$ are all differentially related. This defines an equivalence relation. The union of the elements of an equivalence class $[\mathcal{U}]$ gives a maximal atlas, i.e. a differentiable structure on M.

Def. An n-dimensional manifold is a Hausdorff, second-countable topology on a set M and an n-dim'l differentiable structure on M.