HW 12, due noon Thursday, May 16 *Math 411*

1. Give geometric explanations for the following calculations in $A^*\mathbb{G}_1\mathbb{P}^3$:

   (a) $(0,3)^2 = (1,2)^2 = (0,1)$;

   (b) $(1,3)(0,3) = (1,3)(1,2) = (0,2)$;

   (c) $(0,3)(1,2) = 0$;

   (d) $(1,3)^2 = (0,3) + (1,2)$.

2. The Chow ring of $\mathbb{G}_1\mathbb{P}^4$.

   (a) List all the Schubert classes for $\mathbb{G}_1\mathbb{P}^4$. For each class, state the codimension, give both the a-notation and λ-notation, and describe the associated Schubert condition. For example, one line of your list will be

codim	class	condition
3	$(1,3) = \{2,1\}$	meet a line lying in a solid

   a "solid" being a 3-dimensional linear subspace.

   (b) Make the 10×10 multiplication table for the Chow ring of $A^*(\mathbb{G}_1\mathbb{P}^4)$.

   (c) Find m so that there will be a finite number of lines meeting m given lines, in general. Then find the number of lines meeting m general lines.

   (d) How many lines will meet 6 planes, in general?

   (e) What is the degree of $\mathbb{G}_1\mathbb{P}^4 \subset \mathbb{P}^9$ (Plücker embedding)? (Hint: The *degree* is the number of times a generic linear space of complementary dimension intersects the set. Since $\mathbb{G}_1\mathbb{P}^4$ has dimension $(r+1)(n-r) = 6$, i.e., codimension 3, the degree is given by the number of times a general solid (3-dimensional subspace, codimension 6) in $\mathbb{P}^9$ meets the Grassmannian. To create this solid, note that the Schubert class of codimension 1 is given by intersecting the Grassmannian with a hyperplane.)

   (f) Create your own enumerative problem that can be answered using this Chow ring.

3. **Linear duality.** It turns out that taking the dual of the short exact sequence of vector spaces,
$$0 \to W \to V \to V/W \to 0$$
gives a short exact sequence:
$$0 \to (V/W)^* \to V^* \to W^* \to 0.$$

This gives rise to an isomorphism of Grassmannians:

$$\begin{array}{rcl} G(r,V) & \to & G(n-r,V*) \\ W & \to & (V/W)^* \end{array}$$

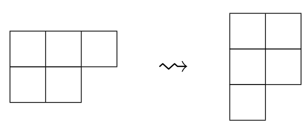
where $n = \dim V$. Fixing an inner product on V identifies V with V^*, and then the isomorphism of Grassmannians is given by sending W to its orthogonal complement. Picking a basis for V, we get $G(r,n) \approx G(n-r,n)$. In projective language,

$$\mathbb{G}_r\mathbb{P}^n = G(r+1,n+1) \approx G(n-r,n+1) = \mathbb{G}_{n-r-1}\mathbb{P}^n.$$

A special case is the duality between points and hyperplanes in $\mathbb{P}^n$ given by

$$(a_0,\dots,a_n) \quad \leftrightarrow \quad a_0x_0 + \cdots + a_nx_n = 0.$$

There is a corresponding duality with the Chow rings for Grassmannians. The dual for the Schubert class $\{\lambda_0,\dots,\lambda_r\}$ in the Chow ring for $\mathbb{G}_r\mathbb{P}^n$ having Young diagram Y_λ is the class in the Chow ring for $\mathbb{G}_{n-r-1}\mathbb{P}^n$ having as its Young diagram the transpose of Y_λ. For instance, the class $\{3,2\} \in A^5\mathbb{G}_1\mathbb{P}^4$ is dual to $\{2,2,1\} \in A^5\mathbb{G}_2\mathbb{P}^4$:



Each of the enumerative problems we have considered using the Schubert calculus has an associated dual problem. For instance, in the plane, there is one line containing two given points and there is one point contained in two lines.

(a) List all the Schubert classes for $\mathbb{G}_2\mathbb{P}^4$, their codimensions, and the relevant Schubert conditions. Match each class with its dual class in the Chow ring for $\mathbb{G}_1\mathbb{P}^4$. (Compare the Schubert conditions for each class and its dual. Note how the word "contains" is dual to "contained in" but the word "meeting" is self-dual.)

(b) What are the duals of the enumerative problems in 2 (c), (d), and (f)?