1. Let $P$ be the hexagon in the plane with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, $(2, 1)$, $(1, 2)$, and $(2, 2)$.

(a) Draw the fan, $\Delta$, for the corresponding toric variety, $X$, labeling the first lattice points along its 1-dimensional cones.

(b) Choose two adjacent 2-dimensional cones in $\Delta$, construct the two corresponding affine toric varieties, and show the gluing instructions.

(c) Show that $X$ smooth.

(d) Calculate the Chow ring, $A^*(X)$.

(e) Calculate the cohomology $H^k X$ for all $k$.

(f) Give the mapping of $X$ into $\mathbb{P}^6$ determined by the 7 lattice points of $P$ (there is one interior point) in homogeneous coordinates.

(g) Pick two components of your mapping, and show that they have the same degree.

(h) Describe $X$ as a quotient, $(\mathbb{C}^{\Delta(1)} \setminus Z)/(x \sim \ell \cdot x)$ according to Cox’s theorem.

2. We have seen that the toric variety $Y$ determined by the single cone, $\sigma = \mathbb{R}_{>0}(2, -1) + \mathbb{R}_{>0}(0, 1)$, has semigroup algebra

$$\mathbb{C}[x, xy, xy^2] \cong \mathbb{C}[u, v, w]/(uw - v^2).$$

Thus, $Y = \{(u, v, w) \in \mathbb{C}^3 : uw = v^2\}$, a cone. The fact that $Y$ has a singularity can be inferred from the cone since

$$\det \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix} = -2 \neq \pm 1.$$

To desingularize $Y$ we “blow-up” the singular point by splitting $\sigma$ into two “nonsingular” cones. Let $\Delta$ be the fan with maximal cones

$$\sigma_1 = \mathbb{R}_{>0}(2, -1) + \mathbb{R}_{>0}(1, 0)$$

$$\sigma_2 = \mathbb{R}_{>0}(1, 0) + \mathbb{R}_{>0}(0, 1),$$

and let $X = X(\Delta)$ be the corresponding toric variety.

(a) Describe how $X$ is obtained from gluing two copies of $\mathbb{C}^2$.

(b) Describe a mapping $\pi : X \to Y$ such that $\pi^{-1}(p)$ consists of a single point for all $p \in Y \setminus \{(0, 0, 0)\}$ and such that $\pi^{-1}(0, 0, 0)$ is a “line”. (Letting $x$ and $y$ be the indeterminates corresponding to the lattice points $(1, 0)$ and $(0, 1)$, respectively, and writing all coordinate functions in terms of $x$ and $y$ should guide the way. Show how the mapping is defined on the two copies of $\mathbb{C}^2$ that are glued to form $X$.)
3. (a) Suppose \( L : V \to W \) and \( M : V' \to W' \) are linear mappings of finite-dimensional vector spaces. There is an induced mapping

\[
L \otimes M : V \otimes V' \to W \otimes W'
\]

\[
v \otimes v' \mapsto L(v) \otimes M(v')
\]

Choosing bases \( v_1, \ldots, v_n \) for \( V \); \( w_1, \ldots, w_m \) for \( W \); \( v'_1, \ldots, v'_t \) for \( V' \); and \( w'_1, \ldots, w'_s \) for \( W' \), we identify \( L \) and \( M \) with matrices. Choosing the corresponding bases

\[
v_1 \otimes v'_1, v_1 \otimes v'_2, \ldots, v_1 \otimes v'_t, v_2 \otimes v'_1, \ldots, v_n \otimes v'_t
\]

for \( V \otimes V' \) and

\[
w_1 \otimes w'_1, w_1 \otimes w'_2, \ldots, w_1 \otimes w'_s, w_2 \otimes w'_1, \ldots, w_m \otimes w'_s
\]

for \( W \otimes W' \) (in the given orders) describe \( L \otimes M \) as a matrix.

(b) With the above conventions, what is \( L \otimes M \) for

\[
L = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.
\]

4. Consider an exact sequence of finite-dimensional vectors spaces:

\[
0 \to V_1 \to V_2 \to \cdots \to V_k \to 0.
\]

Prove that \( \sum_{i=1}^{k} (-1)^i \dim V_i = 0 \).