HW 2, due Friday, February 15 *Math 411*

1. Show that the composition

$$T_p^{\text{phys}} M \to T_p^{\text{geom}} M \to T_p^{\text{alg}} M \to T_p^{\text{phys}} M$$

is the identity.

2. Consider the projective plane $\mathbb{P}^2$ with homogeneous coordinates (x, y, z), and let $p = (1, 1, 1) \in \mathbb{P}^2$. Define

$$f(x, y, z) = \frac{x}{y}.$$

   (a) Show that f is a well-defined function in a neighborhood of the point p.

   (b) Consider the curve in $\alpha(t) = (1+t, 1+t^2, 1+t^3) \in \mathbb{P}^2$ for t in a small open interval about 0. The curve α determines a derivation, v_α, of germs at p. What is $v_\alpha(f)$?

   (c) Consider the standard chart (U_x, ϕ_x) at p, i.e., $U_x = \{(x, y, z) \in \mathbb{P}^2 \mid x \neq 0\}$ with $\phi_x(x, y, z) = (y/x, z/x)$. Let (u, v) denote the coordinates on $\mathbb{R}^2$ here. Fixing this chart gives a basis for $T_p\mathbb{P}^2$ of the form

$$\left(\frac{\partial}{\partial u}\right)_p, \left(\frac{\partial}{\partial v}\right)_p.$$

   What is the tangent vector determined by α in terms of these coordinates?

   (d) Repeat the previous exercise, (2c), with respect to the chart (U_y, ϕ_y).

   (e) Show that your solution to (2c) is sent to your solution to (2d) by the derivative of the change of basis mapping $\phi_y \circ \phi_x^{-1}$.

3. [See exercise 2.3 in our text.] Consider the manifold $M = (0, \infty) \subset \mathbb{R}$. Let $f\colon M \to \mathbb{R}$ be a differentiable mapping, and let $p \in M$. Let $\mathcal{D}_p(M)$ be the collection of charts at p. The gradient relative to each chart gives a mapping

$$\begin{aligned} &\mathcal{D}_p(M) \to \mathbb{R} \\ &(U, h) \mapsto \nabla(f \circ h^{-1})(h(p)) = \frac{d}{dx}(f \circ h^{-1})(h(p)). \end{aligned}$$

Show that this mapping does not, in general, give a tangent vector in $T_p^{\text{phys}} M$. [Explicitly choose a point p, two charts at p, and a function f to illustrate your point.]

4. Is the cross-product mapping

$$\begin{aligned} \omega\colon \mathbb{R}^3 \times \mathbb{R}^3 &\to \mathbb{R}^3 \\ (u, v) &\mapsto u \times v \end{aligned}$$

multilinear and alternating? Explain.