Homology of Simplicial Complexes

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Introduction. This is an introduction to the homology of simplicial complexes suitable for a first course in linear algebra. It uses little more than the rank-nullity theorem. As you read, note how the homology of a simplicial complex drawn on a manifold can measure the number of holes of various dimensions.

Simplicial Complexes

An (abstract) *simplicial complex* Δ on $[n] := \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$, closed under the operation of taking subsets. We will also require that $\{i\} \in \Delta$ for $i = 1, \ldots, n$. The elements of a simplicial complex are called *faces*. An element $\sigma \in \Delta$ of cardinality $i + 1$ is called an *i-dimensional face* or an *i-face* of Δ. The empty set, ϕ, is the unique face of dimension -1. The *dimension* of Δ, denoted $\dim(\Delta)$, is defined to be the maximum of the dimensions of its faces.

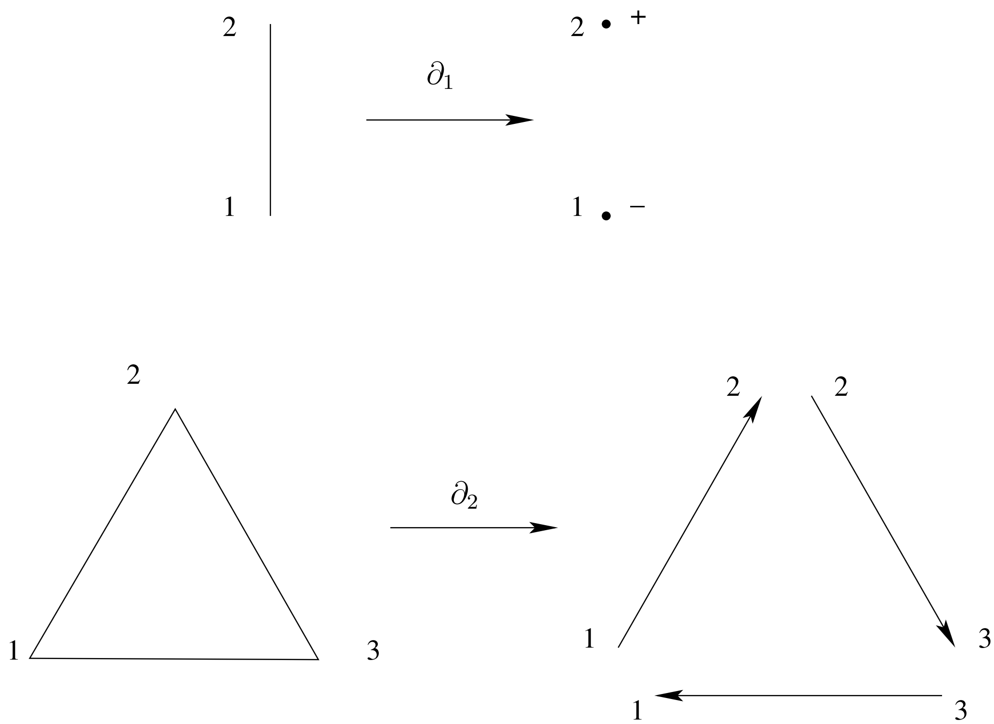
SIMPLICIAL HOMOLOGY. Let Δ be a simplicial complex on $[n]$. For $i \in \mathbb{Z}$, let $F_i(\Delta)$ be the set of i-dimensional faces of Δ, and for each $\sigma \in F_i(\Delta)$, let e_σ denote the corresponding basis vector in the k-vector space, $k^{F_i(\Delta)}$. The *(augmented) chain complex of* Δ *over* k is the complex

$$0 \longrightarrow k^{F_{n-1}(\Delta)} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow k^{F_i(\Delta)} \xrightarrow{\partial_i} k^{F_{i-1}(\Delta)} \longrightarrow \cdots \xrightarrow{\partial_0} k^{F_{-1}(\Delta)} \longrightarrow 0$$

where, for all $i = 0, 1, \ldots, n-1$, and $\sigma \in F_i(\Delta)$,

$$\partial_i(e_\sigma) := \sum_{j \in \sigma} \operatorname{sign}(j, \sigma)\, e_{\sigma \setminus j}.$$

If $i > n - 1$ or $i < -1$, then $k^{F_i(\Delta)} := 0$, and define $\partial_i := 0$. Take $\operatorname{sign}(j, \sigma) = (-1)^{i-1}$ if j is the i-th element of σ when the elements of σ are listed in increasing order. The reader should make the routine check that $\partial_i \circ \partial_{i+1} = 0$. The following pictures provides geometric motivation for the definition of the boundary map:



Boundary maps.

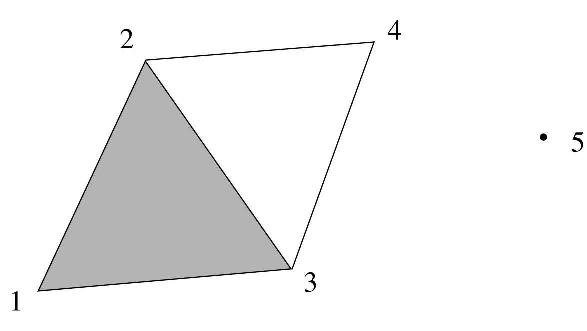
The arrow pointing from 3 to 1, rather than from 1 to 3, geometrically encodes the negative sign in the boundary map: $\partial(e_{\{1,2,3\}}) = e_{\{2,3\}} - e_{\{1,3\}} + e_{\{1,2\}}$.

For $i \in \mathbb{Z}$, the *i-th reduced homology* of Δ over k is the k-vector space

$$\bar{H}_i(\Delta; k) := \text{kernel}(\partial_i)/\text{image}(\partial_{i+1}) \approx k^d,$$

where $d := \dim \text{kernel}(\partial_i) - \dim \text{image}(\partial_{i+1})$. In particular, $\bar{H}_{n-1}(\Delta; k) = \text{kernel}(\partial_{n-1})$ and $\bar{H}_i(\Delta; k) = 0$ for $i > n-1$ or $i < 0$. The dimension of $\bar{H}_0(\Delta; k)$ as a k-vector space is one less than the number of connected components of Δ. Elements of $\text{kernel}(\partial_i)$ are called *i-cycles* and elements of $\text{im}(\partial_{i+1})$ are called *i-boundaries*.

Example. Define the simplicial complex Δ on $[5]$ consisting of all subsets of the sets $\{1,2,3\}$, $\{2,4\}$, $\{3,4\}$, and $\{5\}$, pictured below:



Simplicial complex.

$$\begin{aligned}F_2(\Delta) &= \{\{1,2,3\}\},\\ F_1(\Delta) &= \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\}\},\\ F_0(\Delta) &= \{\{1\},\{2\},\{3\},\{4\},\{5\}\},\\ F_{-1}(\Delta) &= \{\phi\}.\end{aligned}$$

Choosing bases for the $k^{F_i(\Delta)}$ as suggested by the ordering of the faces listed above, the chain complex for Δ becomes

$$0 \xrightarrow[\partial_3]{} k \xrightarrow[\partial_2]{\begin{pmatrix}1\\-1\\1\\0\\0\end{pmatrix}} k^5 \xrightarrow[\partial_1]{\begin{pmatrix}-1&-1&0&0&0\\1&0&-1&-1&0\\0&1&1&0&-1\\0&0&0&1&1\\0&0&0&0&0\end{pmatrix}} k^5 \xrightarrow[\partial_0]{\begin{pmatrix}1&1&1&1&1\end{pmatrix}} k \longrightarrow 0$$

For example, $\partial_2(e_{\{1,2,3\}}) = e_{\{2,3\}} - e_{\{1,3\}} + e_{\{1,2\}}$, which we identify with the vector $(1,-1,1,0,0)$. The mapping ∂_1 has rank 3, so $\bar{H}_0(\Delta;k) \approx \bar{H}_1(\Delta;k) \approx k$ and the other $\bar{H}_i(\Delta;k)$ are 0. Geometrically, $\bar{H}_0(\Delta;k)$ is nontrivial since Δ is disconnected and $\bar{H}_1(\Delta;k)$ is nontrivial since Δ contains a triangle which is not the boundary of an element of Δ.

$\star$ In general, the dimension of $\bar{H}_i(\Delta;k)$ gives the number of $i+1$-dimensional "holes" in the complex (in the sense of a hole made by an i-dimensional object).

Exercises.

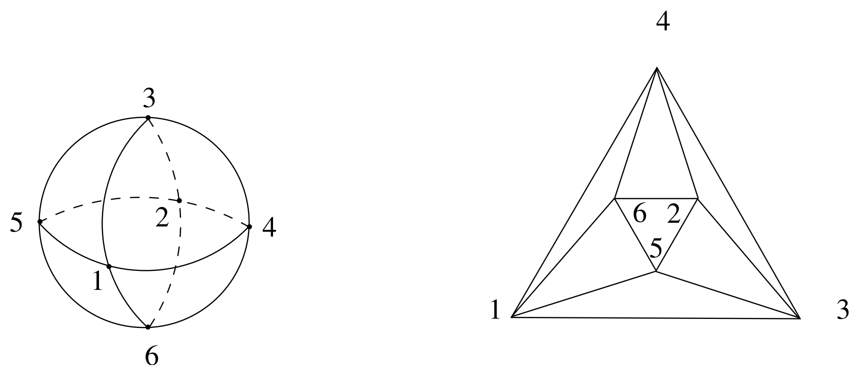
1. Compute the homology for the simplicial complex on [5] consisting of all subsets of the sets $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$, and $\{5\}$. Draw a picture (it's the same as the example given above, omitting the solid triangle $\{1,2,3\}$).

2. Make your own (small-ish) simplicial complex and calculate its homology.

3. Prove that $\partial_i \circ \partial_{i+1} = 0$ for all i.

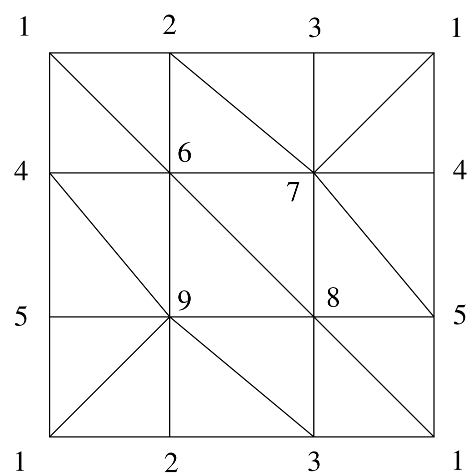
4. Let Δ be the simplicial complex formed by taking all subsets of $[n]$. Describe the chain complex, including the boundary maps, and calculate the homology groups. You might start by doing a few examples for small n. You can prove a general result by induction.

5. Below is a simplicial complex drawn on a sphere. To the right is the same simplicial complex, projected into the face $\{1,3,4\}$. In all, there are 8 two-dimensional faces, counting $\{1,3,4\}$. These are the top-dimensional faces. What are the homology groups for this simplicial complex? How are your results reflected in the geometry of the sphere. (An important fact is that it turns out that you would get the same homology groups for any simplicial complex drawn on the sphere.)



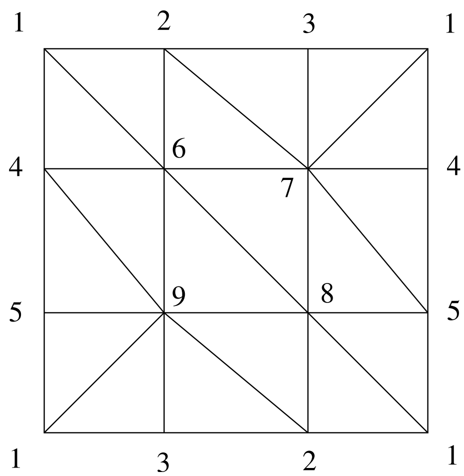
Simplicial complex on a sphere.

6. Below is a simplicial complex with 9 zero-dimensional faces (vertices), 27 one-dimensional faces (edges), and 18 two-dimensional faces. You are supposed to identify (glue) the edges along the top to the corresponding edges on the bottom and the edges along the left side with the corresponding edges on the right side. In this way, it is not hard to see that this simplicial complex can be drawn on the surface of a donut (a torus). Compute the homology groups. How many holes does a (hollow) donut have in each dimension?

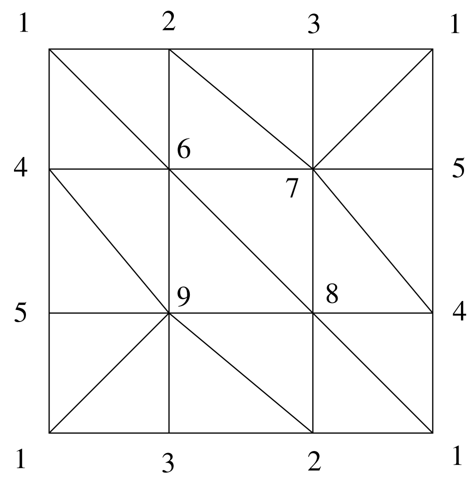


Simplicial complex on a torus.

7. By changing the gluing instructions in the previous exercise, we get simplicial complexes that cover the Klein bottle and the real projective plane, $\mathbb{RP}^2$:



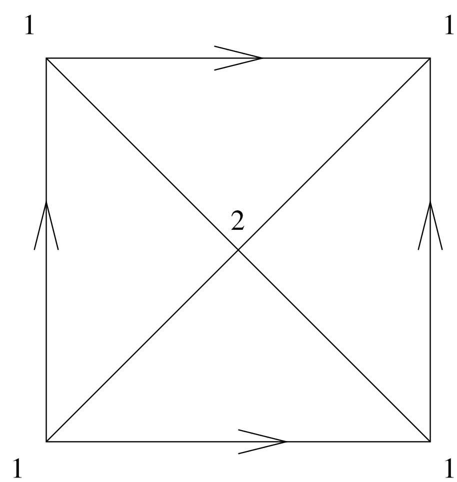
Klein bottle



$\mathbb{RP}^2$

Compute their homology groups and comment on the geometric significance. (For experts: what happens if you work over $\mathbb{Z}$ rather than $\mathbb{R}$?)

8. Can you find a simpler triangulation of the torus? What is wrong with the picture below? (The arrows indicate gluing instructions: glue the top and bottom so that the arrows match, then glue the left and right.)



Triangulation of a torus?