Math 411

$G(r,n)$ = r-dim'l subspaces of k^n; $\mathbb{G}_r\mathbb{P}^n = G(r+1,n+1)$ = r-planes in $\mathbb{P}^n$

Last time: If $k = \mathbb{R}$ or $\mathbb{C}$, then $G(r,n)$ is a manifold.

$\cong \{r\times n \text{ matrices}\} \setminus \{r\times n \text{ matrices s.t. all } r\times r \text{ minor determinants} = 0\} / A \sim MA$ for $r\times r$ invertible M

Today: Embed $G(r,n)$ in projective space

Represent $L \in G(r,n)$ via its homogeneous coordinates, i.e. as an $r\times n$ matrix. For each $j: 1 \le j_1 < \cdots < j_r \le n$, let L_j denote the $r\times r$ submatrix of L formed from columns $j_1, \ldots, j_r$.
Fix an ordering on the $\binom{n}{r}$ choices for j and defined the Plücker embedding:

$$\Lambda : G(r,n) \longrightarrow \mathbb{P}^{\binom{n}{r}-1}$$
$$L \longmapsto (\det L_j)_j$$

Def. These are the Plücker coordinates of L.

Example:

$$\Lambda : G(2,4) \longrightarrow \mathbb{P}^5$$

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix} \longmapsto (a_0b_1 - a_1b_0,\ a_0b_2 - a_2b_0,\ a_0b_3 - a_3b_0,\ a_1b_2 - a_2b_1,\ a_1b_3 - a_3b_1,\ a_2b_3 - a_3b_2)$$

Claim: Λ is well-defined

Pf/ Suppose $r \times n$ matrices A and B both represent L. Then $\exists$ an invertible $r \times r$ matrix M s.t. $B = MA$. For each choice of r columns, j, we have $B_j = MA_j$. Thus, $\det B_j = \det M \det A_j$. Therefore, $(\det B_j)_j = \lambda (\det A_j)_j$, where $\lambda = \det M \neq 0$. Hence,

$$\Lambda(B) = \Lambda(A) \in \mathbb{P}^{\binom{n}{r}-1}. \ \square$$

Coordinate-free description: ← This also shows the map is well-defined.

$G(r,V)$ = r-dim'l subspaces of V

$$\Lambda : G(r,V) \longrightarrow \mathbb{P}(\Lambda^r V) = \text{1-dimensional subspaces of } \Lambda^r V$$
$$W \longmapsto \Lambda^r W$$

Equations defining the image of Λ

Coordinates on $\mathbb{P}^{\binom{n}{r}-1}$: $x(j_1,\dots,j_r)$ with $1 \le j_1 < \dots < j_r \le n$.

Conventions: For a permutation $\sigma \in S_r$ = symmetric group,

$$x(\sigma(j_1),\dots,\sigma(j_r)) := \operatorname{sgn}(\sigma)\, x(j_1,\dots,j_r)$$

$$x(j_1,\dots,j_r) = 0 \quad \text{if} \quad j_s = j_t \quad \text{for some} \quad s \neq t.$$

Def. Plücker relations

For each $I: 1 \le i_1 < \cdots < i_{r-1} \le n$ and $J: 1 \le j_1 < \cdots < j_{r+1} \le n$,

$$P_{I,J} := \sum_{\lambda=1}^{r+1} (-1)^{\lambda} \, x(i_1, \ldots, i_{r-1}, j_\lambda) \, x(j_1, \ldots, \hat{j}_\lambda, \ldots, j_{r+1}).$$

Example For $G(2,4)$, $I: 1$, $J: 2,3,4$

$$P_{I,J} = -x(1,2)x(34) + x(1,3)x(2,4) - x(1,4)x(2,3)$$

In this case, all other choices for I and J yield the same relation, up to sign. It turns out that

$$\Lambda(G(2,4)) = \{x \in \mathbb{P}^5 : P_{I,J}(x) = 0\},$$

a quadric hypersurface in $\mathbb{P}^5$.

Thm. $\Lambda : G(r,n) \to \mathbb{P}^{\binom{n}{r}-1}$ is 1-1 with image defined by the $\{P_{I,J}\}_{I,J}$.

Pf/ We will just show that the image satisfies the equations $P_{I,J} = 0$. Let $L \in G(r,n)$. Then

$$P_{I,J} = \sum_{\lambda=1}^{r+1} (-1)^\lambda \, x(i_1, \dots, i_r, j_\lambda)\, x(j_1, \dots, \hat{j}_\lambda, \dots, j_{r+1}) \implies$$

$$P_{I,J}(\Lambda L) = \sum_{\lambda=1}^{r+1} (-1)^\lambda \det L_{i_1,\dots,i_r,j_\lambda} \det L_{j_1,\dots,\hat{j}_\lambda,\dots,j_{r+1}}$$

$$= \sum_{\lambda=1}^{r+1} (-1)^\lambda \underbrace{\begin{vmatrix} \cdots & a_{1j_\lambda} \\ & \vdots \\ & a_{rj_\lambda} \end{vmatrix}}_{\text{expand}} \begin{vmatrix} \cdots & \hat{a}_{1j_\lambda} & \cdots \\ & \vdots & \\ & \hat{a}_{rj_\lambda} & \end{vmatrix}$$

$$= \pm \sum_{\lambda=1}^{r+1} (-1)^\lambda \sum_{k=1}^{r} (-1)^k a_{kj_\lambda} \begin{vmatrix} \vdots & & \vdots \\ \hat{a}_{ki_1} & \cdots & \hat{a}_{ki_{r-1}} \\ \vdots & & \vdots \end{vmatrix} \begin{vmatrix} \ddots \end{vmatrix}$$

$$= \pm \sum_{k=1}^{r} (-1)^k \left| \ddots \right| \left(\sum_{\lambda=1}^{r+1} (-1)^\lambda a_{k j_\lambda} \begin{vmatrix} & \widehat{a_{1 j_\lambda}} & \\ \cdots & \vdots & \cdots \\ & \widehat{a_{r j_\lambda}} & \end{vmatrix} \right)$$

$$= \pm \sum_{k=1}^{r} (-1)^k \left| \quad \right| \underbrace{\begin{vmatrix} a_{k j_1} & \cdots & a_{k j_\lambda} & \cdots & a_{k j_{r+1}} \\ a_{1 j_1} & \cdots & a_{1 j_\lambda} & \cdots & a_{1 j_{r+1}} \\ & & \vdots & & \\ a_{r j_1} & \cdots & a_{r j_\lambda} & \cdots & a_{r j_{r+1}} \end{vmatrix}}_{0} \begin{array}{l} \leftarrow \text{repeated row} \\ \\ \\ \\ \end{array}$$

$$= 0. \quad \square$$