# Math 411 Grassmannians



**Def.** If $V$ is a vector space over $k$, then $\mathbb{P}(V)$ is the collection of 1-dimensional vector spaces of $V$. A special case is $\mathbb{P}(k^{n+1}) =: \mathbb{P}^n_k$.

**Def.** An r-plane in $\mathbb{P}(V)$ is an $(r+1)$-dimensional subspace of $V$.

**Example** An $(n-1)$-plane in $\mathbb{P}^n$ is the solution set to a linear equation $a_0x_0 + \cdots + a_nx_n \overset{\star}{=} 0$ for some $(a_0, \ldots, a_n) \neq 0$ ($(a_0, \ldots, a_n)$ is the normal vector to the $(n-1)$-plane). Note that the solution set for $\star$ is the same as that for $\lambda a_0 x_0 + \cdots + \lambda a_n x_n = 0$.

We get the duality mapping

$$\mathbb{P}^n \simeq (\mathbb{P}^n)^* = \text{dual projective space} = \{(n-1)\text{-planes of } \mathbb{P}^n\}$$

$$(a_0, \dots, a_n) \longleftrightarrow \{x \in \mathbb{P}^n : a \cdot x = 0\}.$$

**Def.** Let $V$ be a vector space over $k$. Then Grassmannian of $(r+1)$-dimensional subspaces of $V$ is $G(r+1, V)$. It is also called the Grassmannian of $r$-planes in $\mathbb{P}(V)$, then denoted $\mathbb{G}_r \mathbb{P}(V)$. If $V = k^{n+1}$, we write $G(r+1, n+1)$ or $\mathbb{G}_r \mathbb{P}^n$.

**Examples**

$G(1, n+1) = \mathbb{P}^n$ ; $G(2,3) = \mathbb{G}_1 \mathbb{P}^2 = (\mathbb{P}^2)^*$ ; $\mathbb{G}_{n-1} \mathbb{P}^n = (\mathbb{P}^n)^*$

$G(2,4) = \mathbb{G}_1 \mathbb{P}^3$ = lines in 3-space.

## Manifold structure

Given $L \in G(r,n)$, we can write $L = \mathrm{Span}\{a_1, \dots, a_r\}$ for some vectors $a_1, \dots, a_n$. Let $A$ be the matrix whose rows are $a_1, \dots, a_n$. An arbitrary $\mathrm{Span}\{b_1, \dots, b_r\} = L$ iff $\exists$ an invertible $r \times r$ matrix $M$ s.t. $B = MA$, where $B$ is the matrix whose rows are the $b_i$. Thus,

Consider the special case where $r = 1$ to get $\mathbb{P}^n$.

$$G(r,n) = \left\{ \begin{matrix} r \times n \text{ matrices of} \\ \text{rank } r \text{ over } k \end{matrix} \right\} \Big/ A \sim MA$$

for invertible $(r \times r)$-matrices, $M$.

How to get "actual" coordinates rather than "homogeneous" coordinates:

Example

$L = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 2 & 4 & 3 & 1 \end{pmatrix} \in G(2,4)$, a line in $\mathbb{P}^3$ locally parametrized by

$$t \mapsto (1,0,3) + t(2,4,3)$$

Pick 2 linearly independent columns of $L$ and row reduce:

$$\begin{array}{cccc} \star & & & \star \\ \end{array} \qquad \begin{array}{cccc} \star & & & \star \\ \end{array} \qquad \begin{array}{cccc} \star & & & \star \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 3 & 1 \\ 2 & 4 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 4 & -3 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & -4 & 3 & 1 \end{pmatrix}$$

local coordinates for $L$

**Point:** Any matrix representative will have the identical reduced form with respect to the $1^{st}$ and $4^{th}$ columns.

**Notation.** In general, if $L \in G(r,n)$ and $j = (j_1, \dots, j_r)$ with $1 \le j_1 < \cdots < j_r$, let

$$L_j = r \times r \text{ matrix formed by columns } j_1, \dots, j_n \text{ of any matrix representative of } L.$$

**Def.** For each $j$: $1 \le j_1 < \cdots < j_r \le n$, let

$$U_j = \{ L \in G(r,n) : \operatorname{rank} L_j = r \}$$

**Prop.** (1) $U_j$ is well-defined.

(2) The $U_j$ cover $G(r,n)$: $G(r,n) = \bigcup_{j:\, 1 \le j_1 < \cdots < j_r \le n} U_j$

(3) If $k = \mathbb{R}$ or $\mathbb{C}$, then each $U_j$ is an open subset of $G(r,n)$ with its quotient topology.

**Pf/** (1) follows since row operations do not change the rank of a matrix.

(2) is basic linear algebra.

For (3) first a word about the topology of $G(r,n)$. The set of $r \times n$ matrices is $k^{r \times n}$ $(= \mathbb{R}^{rn}$ or $\mathbb{C}^{rn})$, which we give the usual topology. Then $G(r,n) \cong E/_{A \sim MA}$ with the quotient topology: $U \subseteq G(r,n)$ is open iff $\pi^{-1}(U)$ is open in $k^{r \times n}$ where $\pi: E \to G(r,n)$ is the canonical projection.

$E :=$ $r \times n$ matrices of rank $r$ thought of as a subset of $k^{rn}$ w/ subspace topology

If $M$ is an $r \times r$ matrix, then $\operatorname{rank}(M) < r$ iff $\det M = 0$.
Thus $\{ M \in k^{r\times r} : \operatorname{rank} M < r \} = \det^{-1}(0)$ is a closed subset of $k^{r\times r}$. (Note: $\det : k^{r\times r} \to k$ is continuous — in fact it's a polynomial. Then, since $\{0\} \subseteq k$ is closed, $\det^{-1}(0)$ is closed.)

Let $p : k^{r\times n} \to k^{r\times r}$ be the projection defined by $p(A) = A_j$.
We have the following composition of continuous maps:

$$\begin{array}{ccccc} E & \xrightarrow{p} & k^{r\times r} & \xrightarrow{\det} & k \\ A & \longmapsto & A_j & & \\ & & M & \longmapsto & \det M \end{array}$$

Let $Z = (\det \circ p)^{-1}(0)$, a closed subset of $k^{r\times n}$. Then $\widetilde{U}_j = E \setminus Z$ is open. Hence, so is $U_j = \widetilde{U}_j / A \sim MA$. $\square$

Also: Note that $U_j \cong k^{r(n-r)}$ $\forall j$.

**Note:** $\dim G(r,n) = r(n-r)$

$$\dim \mathbb{G}_r \mathbb{P}^n = \dim G(r+1, n+1) = (r+1)(n-r)$$

**Example** A point in $U_{124} \subseteq G(3,7)$ has the form

$$r=3 \left\{ \begin{bmatrix} 1 & 0 & * & 0 & * & * & * \\ 0 & 1 & * & 0 & * & * & * \\ 0 & 0 & * & 1 & * & * & * \end{bmatrix} \right.$$

$n-r=4$

where the asterisks are arbitrary.

**Example** $\dim \mathbb{G}_1 \mathbb{P}^3 = \dim (2,4) = 4$. So the collection of lines in 3-space is a 4-dimensional object. Is this reasonable? To determine a line, pick a point, $p$, (3 parameters) and a direction, $v$, ($v \in S^2$, so 2 parameters are required). However, any point along the line can substitute for $p$ (so subtract 1): $3+2-1=4$.